Introduction to analysis on manifolds with corners

Daniel Grieser

(Carl von Ossietzky Universität Oldenburg)

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The concepts and ideas presented in this course were mostly introduced into analysis by R.B. Melrose. The most comprehensive resource is [8], see also [7].

Introductory presentations are given in [2] and [1].

Lectures 1 and 2 were board talks on **manifolds with corners**, **polyhomogeneous functions**, **blow-up** and their use in the analysis of singular problems. The next slide is a version of the table of examples in lecture 2, and after this you find the slides of lecture 3 (on pseudodifferential calculus related to these singular problems). References, including some which are specific to lecture 3, can be found at the end of this file.

Types of degeneration: Examples

Geometric origin	**Vector fields (local basis)**
none (smooth, compact manifold)	∂_{x_i}
infinite cylinder, cone near its tip	$x\partial_x,\ \partial_{y_i}$
cone (e.g. $\mathbb{R}^n$) near infinity	$x^2\partial_x,\ x\partial_{y_i}$
edge or wedge	$x\partial_x,\ x\partial_{y_i},\ \partial_{z_j}$
fibred cusp	$x^2\partial_x,\ x\partial_{y_i}, \partial_{z_j}$
hyperbolic space at infinity	$x\partial_x, x\partial_{y_i}$

The base space for these examples is a manifold with boundary (except in the smooth case), and the local basis refers to a neighborhood of a boundary point, with the boundary defined by $x = 0$. Fibres in the boundary are given by $x = 0$, $y =$ const.

General setup for singular problems (Melrose)

Given: Boundary fibration structure $(X, \mathcal{V})$

X: a compact manifold with boundary (or corners)
$\mathcal{V}$: a Lie algebra of vector fields on X (locally free $C^\infty(X)$ module)

This defines $\mathbf{Diff}^m_{\mathcal{V}}(\boldsymbol{X})$, the $\mathcal{V}$-**principal symbol** and $\mathcal{V}$-**ellipticity** of $A \in \mathrm{Diff}^m_{\mathcal{V}}(X)$, and $\mathcal{V}$-**Sobolev spaces** $H^s_{\mathcal{V}}(X)$.

Goals (elliptic operators):

Construct parametrices of $\mathcal{V}$-elliptic elements of $\mathrm{Diff}^m_{\mathcal{V}}(X)$, up to remainders which are (depending on level of precision required)

- smoothing
- compact
- rapidly vanishing (at the boundary, or at least some faces)

Classical (non-singular) case

X has no boundary, $\mathcal{V} =$ all smooth vector fields on X

General principles for studying singular problems

Preliminary step: Put problem in the form $(X, \mathcal{V})$.
(This may involve blow-ups, e.g. cone $\rightsquigarrow X$)

General principles for studying $(X, \mathcal{V})$

- Split into geometric and analytic aspects:
  - Geometry encodes singular structure
  - Analysis: conormal distributions ('hide' Fourier transform)
- Separate different types of singular behavior by blow-ups
- Describe operators via their Schwartz kernels
- Use **model problems**

The idea of model problems

Constructive approach

1. Solve model problems (= limit problems)
2. Patch model solutions together
3. Justify: Show that we get approximate solution; remove/estimate errors

Non-singular case: $\partial X = \emptyset$, $A = a(p, D_p) \in \mathrm{Diff}^m(X)$ elliptic.

1. Model problems: $A_{p_0} = a_m(p_0, D_p)$, $p_0 \in X$ (**'zoom in'** at p_0) (constant coefficients $\rightsquigarrow$ invert by Fourier transform, get $B_{p_0}(p, p'))$
2. Patch: $B(p, p') := B_p(p, p')$
3. Justify: $AB = I + R$, $\mathrm{ord}(R) = -1$ Pseudodifferential calculus!

Case of conical singularity:

Additional model problem at tip of cone, solved by Mellin transform.
$\rightsquigarrow$ b-calculus

Classical ΨDO calculus

X = compact smooth manifold

Operators	Symbols (on T^*X)	Schwartz kernels (in $\mathcal{D}'(X^2)$)
$\mathrm{Diff}^*(X)$	homog. polynomials in ξ	δ-type at Diag_X
$\Psi^*(X)$	homog. functions in ξ	Conormal w.r.t. Diag_X

- **Composition Theorem:** $\Psi^*(X)$ is closed under products and the symbol map $\sigma_* : \Psi^*(X) \to S^*(T^*X)$ preserves products
- There is a **short exact symbol sequence** $0 \to \Psi^{m-1}(X) \to \Psi^m(X) \to S^{[m]}(T^*X) \to 0$
- Asymptotic completeness

Theorem

These properties give parametrix construction: $A \in \Psi^m(X)$ elliptic $\Rightarrow$ $\exists B \in \Psi^{-m}(X)$ with $AB - I, BA - I \in \Psi^{-\infty}(X)$.

Classical ΨDO calculus

Functional analysis:

- $A \in \Psi^m(X)$ bounded $H^s(X) \to H^{s-m}(X)$
- $R \in \Psi^{-\infty}(X) \Rightarrow K_R$ smooth $\Rightarrow R$ compact operator

Corollary

$A \in \Psi^m(X)$ elliptic, then

- elliptic regularity: $Au = f$, $f \in H^{s-m}(X) \Rightarrow u \in H^s(X)$
- A Fredholm

Note

Trivially extends to systems, i.e. operators $A : C^\infty(X, E) \to C^\infty(X, F)$ for vector bundles $E, F \to X$.

Small $\mathcal{V} - \Psi$DO calculus

$X =$ compact manifold with corners, $\mathcal{V}$ Lie algebra of vector fields

Operators	Symbols (on ${}^{\mathcal{V}}T^*X$)	Schwartz kernels (in $\mathcal{D}'(X^2_{\mathcal{V}})$)
$\mathrm{Diff}^*_{\mathcal{V}}(X)$	homog. polynomials in ξ	δ-type at $\mathrm{Diag}_{X,\mathcal{V}}$
$\Psi^*_{\mathcal{V}}(X)$	homog. functions in ξ	Conormal w.r.t. $\mathrm{Diag}_{X,\mathcal{V}}$

- **Composition Theorem:** $\Psi^*_{\mathcal{V}}(X)$ is closed under products and the symbol map ${}^{\mathcal{V}}\sigma_* : \Psi^*_{\mathcal{V}}(X) \to S^*({}^{\mathcal{V}}T^*X)$ preserves products
- There is a **short exact symbol sequence** $0 \to \Psi^{m-1}_{\mathcal{V}}(X) \to \Psi^m_{\mathcal{V}}(X) \to S^{[m]}({}^{\mathcal{V}}T^*X) \to 0$
- Asymptotic completeness

Theorem

These properties give parametrix construction: $A \in \Psi^m_{\mathcal{V}}(X)$ elliptic $\Rightarrow$ $\exists B \in \Psi^{-m}_{\mathcal{V}}(X)$ with $AB - I, BA - I \in \Psi^{-\infty}_{\mathcal{V}}(X)$.

Small $\mathcal{V} - \Psi$DO calculus

Functional analysis:

- $A \in \Psi^m_{\mathcal{V}}(X)$ bounded $H^s_{\mathcal{V}}(X) \to H^{s-m}_{\mathcal{V}}(X)$
- $R \in \Psi^{-\infty}_{\mathcal{V}}(X) \Rightarrow K_R$ smooth (but $\not\Rightarrow$ R compact operator)

Corollary

$A \in \Psi^m_{\mathcal{V}}(X)$ $\mathcal{V}$-elliptic, then

- 'small' elliptic regularity: $Au = f$, $f \in H^{s-m}_{\mathcal{V}}(X) \Rightarrow u \in H^s_{\mathcal{V}}(X)$

To get compact errors (hence Fredholm A), need **larger calculus** or **stronger ellipticity condition**.

Main steps in building a $\mathcal{V} - \Psi$DO calculus

1. Construct **double space** $X^2_{\mathcal{V}}$. Requirements:
   - Diagonal Diag_X lifts to p-submanifold $\mathrm{Diag}_{X,\mathcal{V}}$
   - For any $V \in \mathcal{V}$, the vector field $V \times 0$ on X^2 lifts smoothly to $X^2_{\mathcal{V}}$
   - These lifts span the normal space to $\mathrm{Diag}_{X,\mathcal{V}}$
2. Define **small $\mathcal{V}$-calculus** $\Psi^*_{\mathcal{V}}(X)$ via Schwartz kernels on $X^2_{\mathcal{V}}$:
   - conormal w.r.t. $\mathrm{Diag}_{X,\mathcal{V}}$ (uniformly to the boundary)
   - vanish to all orders at all faces except those intersecting $\mathrm{Diag}_{X,\mathcal{V}}$
   - symbols are functions on ${}^{\mathcal{V}}T^*X \cong N^* \,\mathrm{Diag}_{X,\mathcal{V}}$

   $\rightsquigarrow$ can invert $\mathcal{V}$-elliptic operators up to **smoothing** errors.
3. Identify obstruction to compactness of smoothing operators.
   $\rightsquigarrow$ normal, indicial operator(s)
4. If needed, enlarge calculus by including inverses of normal operator(s)
   $\rightsquigarrow$ get compact errors

b-calculus

The problem:
X = cpct manifold with boundary, $\mathcal{V}_b = \{\text{vector fields tangent to } \partial X\}$
(spanned by $x\partial_x, \partial_{y_i}$ near boundary)

The solution:

1. **Double space:** $X_b^2 := [X^2, (\partial X)^2]$
2. **Model operator at boundary:** $I_P(\tau) \in \mathrm{Diff}^m(\partial X)$
(freeze coeff. at boundary, $x\partial_x \rightsquigarrow \tau$)
3. **Small b-calculus:** $\Psi_b^*(X)$, **full b-calculus:** $\Psi_b^{*,\mathcal{E}}(X)$

Simple example

$A = x\partial_x + c$ on $X = \mathbb{R}_+ = [0, \infty)$.
(only analyze behavior near $x = 0$)
Kernels of inverses: $K_B(x, x') = \left(\frac{x'}{x}\right)^c (H(x - x') + \mathrm{const})$

Note: different kinds of singular behavior of K_B are separated

Fibred boundary (φ-calculus)

The problem:
X = cpct manifold with boundary, fibration $Z \to \partial X \xrightarrow{\varphi} Y$
$\mathcal{V}_\varphi$ spanned by $x^2\partial_x, x\partial_{y_i}, \partial_{z_j}$ near boundary (tangent to fibres)

The solution:

1. **Double space:** $X^2_\varphi := [X^2_b, \Delta_\varphi]$, Δ_φ = fibre diagonal
2. **Model operator at boundary:** $N_P(\xi,\eta) \in \mathrm{Diff}^m(Z)$ (freeze coeff. at boundary, $x^2 D_x \rightsquigarrow \xi$, $xD_y \rightsquigarrow \eta$)
3. **Small φ-calculus:** $\Psi^*_\varphi(X)$, **full φ-calculus:** $\Psi^{*,\mathcal{E}}_\varphi(X)$

Example $X = B \times Z$, product metric

$\Delta \approx (x^2 D_x)^2 + (xD_y)^2 + D_z^2$

- On $C^\infty(B,\mathcal{K})$, $\mathcal{K} = \ker D_z^2$, this is x^2 times a b-operator
- On $C^\infty(B,\mathcal{K}^\perp)$, N_P is invertible, hence parametrix in small φ-calculus

Some references I

[1] D.Grieser, *Basics of the b-calculus*, arXiv math.AP/0010314, 2000.

(Appeared in J.B.Gil et al. (eds.), Approaches to Singular Analysis, 30-84, Operator Theory: Advances and Applications, 125. Advances in Partial Differential Equations, Birkhäuser, Basel.)

(leisurely elementary introduction to manifolds with corners, blow-ups and the b-calculus)

[2] D. Grieser, *Scales, blow-up and quasimode constructions,* arXiv math.SP/1607.04171, 2016.

(introduction to mwc and blow-ups with a different outlook than [**?**, Gri:BBC]

[3] D. Grieser, E. Hunsicker, *Pseudodifferential operator calculus for generalized Q-rank 1 locally symmetric spaces, I*, Journal of Functional Analysis, 2009.

(generalizes [6] to the case of several stacked fibrations)

Some references II

[4] D. Grieser, E. Hunsicker, *A Parametrix Construction for the Laplacian on Q-rank 1 Locally Symmetric Spaces*, Proceedings of the Workshop on Fourier Analysis and Pseudo-Differential Operators, Aalto, Finland. Trends in Mathematics, Birkhäuser, Basel 2014.

(φ-calculus for Dirac and Laplace operator in the presence of fibre-harmonic forms at the boundary)

[5] T. Hausel, E. Hunsicker, and R. Mazzeo, *Hodge cohomology of gravitational instantons*, Duke Math. J., 122(3):485–548, 2004.

(computation of L^2-cohomology of fibred cusp and fibred boundary metrics using results from [11])

[6] R. Mazzeo and R. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries* in *"Mikio Sato: a great Japanese mathematician of the twentieth century."*, Asian J. Math. **2** (1998) no. 4, 833–866.

(small ΨDO calculus for fibred cusp operators: $x^2\partial_x, x\partial_y, \partial_z$)

Some references III

[7] R.B.Melrose, *Pseudodifferential operators, corners and singular limits*, Proc. Int. Congr. Math., Kyoto/Japan 1990, Vol. I, 217-234 (1991).
(introduction of a general framework for singular analysis, with examples)

[8] R. Melrose, *Differential analysis on manifolds with corners*, in preparation, partially available at `http://www-math.mit.edu/~rbm/book.html`.
(the details for [7], work in progress)

[9] R. Melrose, *The Atiyah-Patodi-Singer index theorem*, A.K. Peters, Newton (1991).
(detailed introduction of the b-ΨDO calculus, $x\partial_x, \partial_y$ – elliptic and heat kernel parametrix – and application to index theory)

[10] B-W. Schulze, *Boundary Value Problems and Singular Pseudo-Differential Operators*, John Wiley & Sons, (2008).
(another approach to a ΨDO calculus for cone and edge singularities, including boundary value problems)

Some references IV

[11] B. Vaillant, *Index and spectral theory for manifolds with generalized fibred cusps*, Ph.D. thesis, Univ. of Bonn, 2001. arXiv:math-DG/0102072.

(extends the parametrix construction of [6] to the case of non-invertible normal operator, in case of the Dirac operator; also heat kernel and application to index theory)