# London Summer School and Workshop: The Sen Conjecture and Beyond

June 19 -- 23, 2017

*Outlines and Abstracts*

## Lectures

These are coarse outlines for the lecture courses. They might be updated closer to the workshop. (*Last update: 05/06/2017*)

### Daniel Grieser: Introduction to Analysis on Manifolds with Corners

Lecture 1:
Introduction of basics on manifolds with corners, blow-ups, polyhomogeneous functions (with elementary examples and motivations)

Lecture 2:
More of the same, principles for building a calculus (Schwartz kernels, idea of boundary fibration structure, example: b-calculus)

Lecture 3:
Fibered boundaries and the phi-calculus

### Pierre Albin: $L^2$ Cohomology, Stratified Spaces and Related Topics

Lecture 1:
$L^2$ cohomology and Hilbert complexes

Lecture 2:
Some $L^2$ cohomologies on manifolds with fibred boundary (complete and incomplete case)

Lecture 3:
Some $L^2$ cohomologies on stratified spaces, wedge metrics and fibred cusp metrics

### Richard Melrose: Analysis on Planar Hilbert Schemes

I will use the example of the Hilbert schemes of k points in $\mathbb{C}^2$ to describe an approach to the analysis of geometric operators on moduli spaces and other settings with iterative regularity.

Note: I will rely in part on results from the courses of Pierre Albin and Daniel Grieser and will in turn try to indicate the relationship to some of the other lectures. More detailed notes should be available by the beginning of the meeting.

Lecture 1: Compactification of Hilbert schemes
After describing the main result, I will introduce the Hilbert schemes of points in the plane and the Hilbert-Chow maps to the corresponding configuration spaces, then discuss the resolution of the latter. Next I will briefly introduce Nakashima's quiver varieties and hyper-Kähler reduction, leading to

the compactification of the planar Hilbert schemes (or “QALE” structure) by resolution of the group action.

Lecture 2: Iterated fibrations and metrics

In the context of manifolds with corners I will describe classes of metrics corresponding to iterated fibrations of the boundary hypersurfaces, including the hyper-Kähler metric on the compactified Hilbert schemes and iterated edge metrics. The associated generalized products allow the construction of a parametrix for the Hodge operator and I will describe both the geometric and analytic constructions at least in the case of primary interest.

Lecture 3: Hodge and $L^2$ cohomology

In the last lecture I will apply these constructions to the description of the Hodge cohomology of the Hilbert schemes -- with the result predicted by Vafa and Witten -- and in particular describe Hitchin's Lefschetz construction which shows that most of it is trivial. I will end by trying to describe some open problems, of which there are many!

## Andy Royston: Supersymmetric Gauge Theory, Monopole Moduli Space, and “No Exotics” as a Generalized Sen Conjecture

Recent developments in quantum supersymmetric gauge theory have implications for the $L^2$ cohomology of families of (twisted) Dirac-Dolbeault operators on monopole moduli space. Wall-crossing formulae for BPS states lead to predictions for where the Dirac operators fail to be Fredholm, and how their kernels jump, as parameters of the family are varied. The no-exotics property of BPS states leads to a generalization of the Sen conjecture: All non-trivial $L^2$ cohomology of the Dolbeault operators is concentrated in the middle anti-holomorphic degree.

The aim of these lectures is to give a pedagogical description of how these predictions arise from supersymmetric gauge theory. The following is a preliminary outline.

Lecture 1: Classical gauge theory and monopoles

- Yang-Mills-Higgs theory: action, energy functional; Bogomolny bound; monopoles and monopole moduli spaces $\mathcal{M}$
- $\mathcal{N}$=2 super-Yang-Mills: fermions; action, energy functional; (super)symmetry algebra, $su(2)_R$; dyons
- (supersymmetric) ‘t Hooft defects: modified Bogomolny bound; singular monopoles and their moduli spaces

Lecture 2: Quantum super-Yang-Mills and Dirac operators on monopole moduli spaces

- Elements of QFT: Hilbert space of states; Hamiltonian operator and grading of the spectrum; BPS states
- The Seiberg-Witten approach: low-energy effective/Wilsonian approach; constraints from (super)symmetry → SW solution; spectrum of BPS states: wall-crossing formulae and no-exotics
- The semiclassical approach: perturbation theory in each topological component of configuration space; soliton sectors → (supersymmetric) quantum mechanics on $\mathcal{M}$; BPS states ↔ kernel of $\not{D}$
- The Seiberg-Witten--semiclassical map

Lecture 3: Predictions from physics for $L^2$ cohomology

- P1: Generalised Sen from no-exotics: $su(2)_R$ action on $\mathcal{S}_{\text{Dirac}}(\mathcal{M})$; $su(2)_R$ as anti-holomorphic Lefschetz sl(2)
- P2: Fredholm and jumping properties from WCF

- Example: $\mathcal{M}$ = Taub-NUT
- Road map to proving P2: monopole clusters and partitions of charge; boundary faces of $\mathcal{M}$; asymptotic analysis of $\not{D}$

References: arXiv:1512.08924 [hep-th], arXiv:1512:08923 [hep-th], arXiv:1610.00697 [hep-th], arXiv:1512.02979 [math-DG]

## Research Talks

### Francesco Bei: On the $L^2$-$\overline{\partial}$ Cohomology of Certain Complete Kähler Metrics

Saper metrics are an interesting class of complete Kähler metrics of finite volume introduced by L. Saper on the regular part of a complex projective variety V with isolated singularities. Their importance, as proved by Saper, lies in the fact that the $L^2$-de Rham cohomology of reg(V), the regular part of V, is isomorphic to the middle perversity intersection cohomology of V while the $L^2$-$\overline{\partial}$ cohomology of reg(V) is isomorphic to the Dolbeault cohomology of any resolution of V. Subsequently the construction of Saper metrics has been generalized by Grant Melles and Milman to the case of an arbitrary analytic subvariety Z of a compact Kähler manifold M.

The goal of this talk is to describe a recent joint work with Paolo Piazza concerning the $L^2$-$\overline{\partial}$ cohomology of a class of complete Kähler metrics lying on the regular part of a compact complex space and that includes, for instance, the case of Saper type Kähler metrics. More precisely, given a compact and irreducible complex space X whose regular part carries a complete Hermitian metric h, we provide a criterion that assures the existence of the following isomorphism:

$$H^{0,q}_{2,\bar{\partial}}\big(\mathrm{reg}(X),h\big) \cong H^{0,q}_{\bar{\partial}}\big(M\big)\ ,$$

where $\pi$: M $\to$ X is any resolution of X. Finally we will discuss some applications of our result. Besides the case of Saper type Kähler metrics we will show that our criterion applies also in the setting of complete Kähler metrics with finite volume and pinched negative sectional curvatures.

*This is a joint work with Paolo Piazza.*

### Harry Braden: On the Construction of Monopoles

Although the study of BPS monopoles is now over 30 years old there are still few analytic results known for the Higgs and gauge fields. For SU(2)-monopoles without spherical or axial symmetry the only known results are for the Higgs field on a coordinate axis for charge 2. By combining integrable systems and twistor constructions we show how the problem becomes algebraic in an appropriate gauge. In pursuing this programme a number of new results have been found. The approach will be illustrated by completely solving the general charge 2 cases (giving the fields and corresponding energy density).

### Sergey Cherkis: Yang-Mills Instantons on Gravitational Instantons: Index and Monads

The construction of Yang-Mills anti-self-dual connections with square-integrable curvature on a noncompact hyper-Kähler manifold is strongly dependent on the manifold's asymptotic form. We explore the general scheme of such a construction in terms of the index bundle of certain Dirac-type operators and in terms of monads.

### Anda Degeratu: Gluing Techniques for Metrics

Gluing techniques have been establishing themselves as a standard, albeit difficult, approach for obtaining geometrical objects with special properties. In this talk, I will focus on this technique in the context of Kähler metrics which are Ricci-flat, or more generally, have constant scalar curvature. The first requirements for its success are to have a large enough pool of model metrics and to figure out the obstructions for gluing them together into metrics with desired properties.

### Laura Fredrickson: The Ends of the Hitchin Moduli Space

Hitchin's equations are a system of gauge theoretic equations on a Riemann surface that are of interest in many areas including representation theory, Teichmüller theory, and the geometric Langlands correspondence. In this talk, I'll describe what solutions of SL(n,$\mathbb{C}$)-Hitchin's equations "near the ends" of the moduli space look like, and the resulting compactification of the Hitchin moduli space. Wild Hitchin moduli spaces are an important ingredient in this construction. This construction generalizes Mazzeo-Swoboda-Weiss-Witt's construction of SL(2,$\mathbb{C}$)-solutions of Hitchin's equations where the Higgs field is "simple."

### Jesse Gell-Redman: Hodge Theory on the Moduli Space of Riemann Surfaces

We will discuss work toward extending the Hodge theorem to singular Riemannian spaces where the singular locus is locally a product of incomplete cusp edges. These can be pictured locally as products of bundles of geometric horns, and they arise in particular as Weil-Petersson geometry on the compactified Riemann moduli space.
*This talk is based on joint works with Richard Melrose and with Jan Swoboda.*

### Derek Harland: Monopole Chains with Cyclic Symmetry

Monopole chains are monopoles invariant under a discrete 1-dimensional translation group; equivalently, they are monopoles on the manifold $\mathbb{R}^2\times\mathbb{S}^1$. A classical problem for monopoles is the determination of allowed symmetry groups; in this talk I will discuss a classification of monopole chains with cyclic symmetry. The classification depends on much of the technology that has been developed to analyse monopole chains, including spectral curves, the Nahm transform, Hitchin systems, and Higgs bundles.

### Nicholas Manton: Exotic Vortices and their Dynamics

The usual Abelian Higgs Vortex equations can be generalised and unified to include five distinct vortex types. I discuss the integrability of these equations and some explicit solutions in curved space. Often, the moduli space of these vortices is not compact, and the natural Lagrangian dynamics of these vortices is rather exotic, because their kinetic energy is not always positive definite. The consequences are not yet clear.

### Stephen McKeown: Cornered Asymptotically Hyperbolic Einstein Metrics

This talk will concern cornered asymptotically hyperbolic spaces, which have a finite boundary in addition to the usual infinite boundary. After introducing the setting, I will present a normal form near the corner for these spaces, and briefly discuss the geodesic analysis at the corner that provides it. Using this, I will then discuss formal existence and uniqueness, near the blown-up corner, of asymptotically hyperbolic Einstein metrics, with a CMC-umbilic condition imposed on the finite boundary. I will finally identify a conformal hypersurface invariant obstructing smoothness of such a metric with totally geodesic finite boundary.
*This is doctoral work under C. Robin Graham.*

### Lukas Müller: Extended Field Theories from Index Theory

A d-dimensional (quantum) field theory assigns Hilbert spaces of wave functions to closed (d-1)-dimensional manifolds and linear (time evolution) operators to d-dimensional manifolds. In an extended theory, one also assigns categories of boundary conditions to (d-2)-dimensional manifolds. So called invertible extended field theories in d dimensions encode the information about anomalies in (d-1)-dimensional theories.
In my talk, I will present a concrete construction for invertible extended field theories in even dimensions based on Loya's and Melrose's index theorem for manifolds with corners. These theories can be used to describe the parity anomaly of a fermionic system in d-1 dimensions.

### Akos Nagy: Construction of Monopoles with Non-Maximal Symmetry Breaking

The notion of “broken symmetry” is central in gauge theories. For SU(N)-monopoles, symmetry breaking can be defined in terms of the eigenvalues of the Higgs-field at infinity. The symmetry breaking is maximal if the eigenvalues are distinct. Monopoles with maximal symmetry breaking have been studied by both mathematicians and physicists for decades now, but little is known about the general case. I will show how to construct monopoles with arbitrary symmetry breaking using the Nahm transform.

*This is a joint project with Benoit Charbonneau.*

### Gonçalo Oliveira: G2-Instantons on Non-Compact G2-Manifolds

I will report on joint work with Jason Lotay on some existence and non-existence results for G2-instantons. I shall compare the behavior of G2-instantons for two distinct G2-metrics on $\mathbb{R}^4 \square \mathbb{S}^3$.

*This is on a joint project with Jason Lotay.*

### Frédéric Rochon: QAC Calabi-Yau Manifolds

We will explain how to construct new examples of quasi-asymptotically conical (QAC) Calabi-Yau manifolds that are not quasi-asymptotically locally Euclidean (QALE). Our strategy consists in introducing a natural compactification of QAC-spaces by manifolds with fibred corners and to give a definition of QAC-metrics in terms of a natural Lie algebra of vector fields on this compactification. Using this and the Fredholm theory of Degeratu-Mazzeo for elliptic operators associated to QAC-metrics, we can in many instances obtain Kahler QAC metrics having Ricci potential decaying sufficiently fast at infinity. We can then obtain QAC Calabi-Yau metrics in the Kähler classes of these metrics by solving a corresponding complex Monge-Ampère equation.

*This is a joint work with Ronan Conlon and Anda Degeratu.*

### Nuno Romao: Asymptotic Geometry and Topology of BPS-Vortex-Antivortex Moduli Spaces

The vortex equations are the self-duality equations of two-dimensional Yang-Mills-Higgs theory, defining objects analogous to the BPS monopoles entering the Sen conjecture. When the target involves a nonlinear action, BPS vortex configurations may include different types of particles - the simplest case is $\mathbb{S}^2$ with its usual circle action, for which there are two types of particles (“vortices” and “antivortices”) with cores at the zeroes and poles of a meromorphic Higgs field. The underlying moduli spaces are Kähler and have boundaries corresponding to coalescence of zeros with poles. Unless the zeros and poles are coprime in number, their braiding generates nontrivial commutators in the fundamental group of the moduli space (on a surface of positive genus).

I will report on recent progress in these topics, which should be regarded as puzzle pieces to assemble into an appropriate version of the Sen conjecture describing the analytic $L^2$-Betti numbers of these moduli spaces in relation to the representation varieties of their fundamental groups.

### Calum Ross: Magnetic Zero Modes and Vortices

I will mention several ways to construct zero modes for a Dirac operator, twisted by an abelian potential, in three dimensions. These zero modes can be related to a variety of topics: the higher angular momentum zero modes of Loss and Yau and Erdös and Solovej, integrable vortex equations in two dimensions and dimensional reductions of the Seiberg-Witten equations to three dimensions. The key to these relations are vortex zero modes of the Dirac equation on the three sphere and the "vortex equations" they satisfy there.

### Bernd Schroers: Spectral Properties of Gravitational Instantons

Several gravitational instantons support self-dual and square-integrable two-forms generalising Sen's form on the Atiyah-Hitchin manifold. The spectrum of Laplace operators on these spaces twisted by a U(1)-connection with curvature proportional to such a generalised Sen form plays a role in several distinct parts of mathematical physics - ranging from moduli space quantisation of monopoles to semiclassical quantum gravity.

In this talk I discuss the common and special features of these operators for the Taub-NUT space, the Atiyah-Hitchin manifold and the Euclidean Schwarzschild space. These include many of features familiar from standard quantum mechanics in two and three dimensions, including Coulombic bound states, Landau levels and resonances.

### Jan Swoboda: Asymptotic Geometry of the Hitchin Moduli Space

In this talk I will explain recent joint work with Rafe Mazzeo, Hartmut Weiss and Frederik Witt on the asymptotics of the natural $L^2$-metric on the Hitchin moduli space of rank-2 Higgs bundles. It will be shown that on the regular part of the Hitchin fibration this metric is well-approximated by the so-called semi-flat metric coming from the algebraic completely integrable system the moduli space is endowed with. This result confirms some aspects of a more detailed conjectural picture made by Gaiotto, Moore and Neitzke. Part of the results are based on an understanding of the transition between smooth and singular geometric operators which takes place in the limit of large Higgs fields, and I will pay particular attention to these more analytic aspects of the project.

*This is a joint project with Rafe Mazzeo, Hartmut Weiss and Frederik Witt.*

### Xuwen Zhu: The Configuration Space of Constant Scalar Curvature Metrics with Conical Singularities

For a compact Riemann surface M, we would like to understand the space of constant curvature metrics with prescribed conical singularities. When all the cone angles are less than $2\pi$, the existence and uniqueness is known due to the works of Luo-Tian, McOwen and Troyanov; the recent work of Mazzeo and Weiss gives the deformation theory and constructs the moduli space of such metrics. However when some or all of the cone angles are bigger than $2\pi$, at least when the curvature is positive, the analysis is much more complicated, which is suggested by the recent breakthrough of Mondello and Panov by synthetic geometry method that there are global constraints on the space of cone angles.

We discover that one key ingredient of the obstructed deformation is related to allow some of the points to split into clusters with smaller cone angles. We construct a resolution of the configuration space of conical metrics, and prove a new regularity result that the family of hyperbolic or flat metrics with conic singularities has a nice compactification as the cone points coalesce, and moreover, the fibrewise family of constant curvature metrics is polyhomogeneous on this compactification. And we hope to apply this new construction to describe the moduli space of spherical conic metrics with all possible cone angles.

*This is joint work in progress with Rafe Mazzeo.*