BOUNDARY VALUE PROBLEMS FOR DIRAC-TYPE OPERATORS AND AN APPLICATION

PENGSHUAI SHI

Abstract. We introduce briefly the boundary value problems for first order elliptic operators on complete manifolds with compact boundary following the work of [4], [5]. To be precise, we focus on the case where the operator is Dirac-type. The idea is to define the boundary conditions as closed subspaces of a hybrid Sobolev space on the boundary induced by an adapted operator. Taking local H^1-regularity into consideration, one can further define elliptic boundary conditions. This generalizes the Atiyah-Patodi-Singer boundary condition and accounts for nice properties such as regularity and Fredholmness. As an application, we show that the Callias index theorem still holds on manifolds with boundary.

1. Introduction

We consider the boundary value problems for Dirac-type operators. (Actually the theory still holds for certain general first-order elliptic operators). The underlying manifold may be noncompact, but the boundary is assumed to be compact. People would hope to impose certain condition on the boundary so that the solution to the problem has nice properties, such as finite-dimensionality and regularity. One natural requirement of the domain on the boundary is that all elements vanish on it, which implies the minimal extension of the operator. The other one is to impose no restrictions on the boundary, which implies the maximal extension. Generally speaking, the first condition is too strong, whereas the second one is too weak. So one needs to study the conditions lying in between. This is the intuition of identifying boundary conditions as closed extensions between minimal and maximal extensions.

For a Dirac-type operator, one can easily get an adapted operator on the boundary which is formally self-adjoint (cf. Section 3). Using the closedness of the boundary, one has spectral decomposition of the space of square-integrable sections on the boundary induced by an adapted operator. From this, we define a hybrid Sobolev space on the boundary. By the main result Theorem 4.4, boundary conditions are equivalent to closed subspaces of that Sobolev space. Thus we get a concrete way to describe boundary conditions.

One issue on manifolds with boundary is that the elements in the maximal domain may not be locally H^1, which is an obstruction to get regularity and Fredholmness. As a result, we further define elliptic boundary condition requiring local H^1-regularity for the domain. This is more or less an analogue of operator ellipticity on manifolds without boundary. Therefore, we have similar result saying that operators (together with their formal adjoints) which are invertible at infinity are Fredholm. As an enlightening example, we show that the Atiyah-Patodi-Singer boundary condition fits into this theory. We then apply this condition to Callias-type operators to get the Callias index theorem.

This article is mostly a survey with some personal explorations of the theory and application. It follows closely the work of [4], [5]. To avoid technical details, proofs of some main results are omitted or sketched. The article is organized as following.

In Section 2, we talk about basic setting on manifolds with boundary. In Section 3, we introduce Dirac-type operators and construct their adapted operators on the boundary from the principal symbol level. In Section 4, we give important properties of the maximal domain and formally define boundary conditions and elliptic boundary conditions. Then we study the example of APS boundary condition. In Section 5, we do an index theoretical preparation. In Section 6, we apply the APS boundary condition to Callias-type operators and get the Callias index theorem as an immediate consequence.

2. Basic setting on manifolds with boundary

2.1. **Some preliminaries.** We use the notations in [5]. Let M be a complete Riemannian manifold with compact boundary ∂M and volume elements dV on M, dS on ∂M. The interior of M is denoted by $\mathring{M}$. For a vector bundle E over M, $C^\infty(M,E)$ is the space of smooth sections of E, $C_c^\infty(M,E)$ is the space of smooth sections of E with compact support, and $C_{cc}^\infty(M,E)$ is the space of smooth sections in E with compact support in $\mathring{M}$. Note that

$$C_{cc}^\infty(M,E) \subset C_c^\infty(M,E) \subset C^\infty(M,E).$$

And when M is compact, $C_c^\infty(M,E) = C^\infty(M,E)$; when $\partial M = \emptyset$, $C_{cc}^\infty(M,E) = C_c^\infty(M,E)$. We also have $L^2(M,E)$, the Hilbert space of square-integrable sections of E, which is the completion of $C_c^\infty(M,E)$ with respect to the norm induced by the L^2-inner product

$$(u_1,u_2) \;=\; \int_M \langle u_1,u_2\rangle dV,$$

where $\langle\cdot,\cdot\rangle$ denotes the fiberwise inner product.

Let E,F be two Hermitian vector bundles over M and $D : C^\infty(M,E) \to C^\infty(M,F)$ be a first-order differential operator. The *formal adjoint* of D, denoted by D^*, is defined by

$$\int_M \langle Du,v\rangle dV \;=\; \int_M \langle u,D^*v\rangle dV,$$

for all $u \in C_{cc}^\infty(M,E)$ and $v \in C^\infty(M,F)$. If $E = F$ and $D = D^*$, then D is called *formally self-adjoint*. Let σ_D be the principal symbol of D defined by

$$D(fu) \;=\; fDu + \sigma_D(df)u,$$

for all $f \in C^\infty(M,\mathbb{R})$ and $u \in C^\infty(M,E)$. Then

$$\sigma_{D^*}(\xi) \;=\; -\sigma_D(\xi)^*, \tag{2.1}$$

for all $\xi \in T^*M$.

Let $\tau \in TM|_{\partial M}$ be the unit inward normal vector field along ∂M. Using the Riemannian metric, τ can be identified with its associated one-form. We have the following formula (cf. [6, Proposition 3.4]).

Proposition 2.2 (Green's formula). *Let D be as above. Then for all $u \in C_c^\infty(M,E)$ and $v \in C_c^\infty(M,F)$,*

$$\int_M \langle Du,v\rangle dV \;=\; \int_M \langle u,D^*v\rangle dV \;-\; \int_{\partial M} \langle \sigma_D(\tau)u,v\rangle dS.$$

2.3. **Minimal and maximal extensions.** Suppose $D_{cc} := D|_{C_{cc}^\infty(M,E)}$, viewed as an unbounded operator from $L^2(M,E)$ to $L^2(M,F)$. The *minimal extension* $D_{\min}$ of D is the operator whose graph is the closure of that of D_{cc}. The *maxmal extension* $D_{\max}$ of D is the extension of D to $\operatorname{dom} D_{\max}$, which consists of all $u \in L^2(M,E)$ with $Du \in L^2(M,F)$ in the distributional sense. Equivalently, that means, $D_{\max} = (D_{cc}^*)^*$. Both $D_{\min}$ and $D_{\max}$ are closed operators. Their domains, $\operatorname{dom} D_{\min}$ and $\operatorname{dom} D_{\max}$, become Hilbert spaces equipped with the *graph norm*, which is the norm associated the inner product

$$(u_1, u_2)_D \ := \ \int_M (\langle u_1, u_2\rangle + \langle Du_1, Du_2\rangle) dV.$$

Later when we talk about boundary value problems, we mean closed operators lying between $D_{\min}$ and $D_{\max}$.

2.4. **Sololev spaces.** Now in addition, we assume that E and F are endowed with Hermitian connections. Taking E as example. Denote the connection by ∇. For any $u \in C^\infty(M, E)$, the covariant derivative $\nabla u \in C^\infty(M, T^*M \otimes E)$. Then we define the Sobolev space

$$H^1(M, E) \ := \ \{u \in L^2(M, E) : \nabla u \in L^2(M, E)\}$$

again in distributional sense. It is a Hilbert space with Sobolev norm

$$\|u\|^2_{H^1(M)} \ := \ \|u\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)}.$$

Note that when M is compact, $H^1(M, E)$ does not depend on the choices of ∇ and Riemannian metric, but when M is noncompact, it does.

If restricting everything to a compact subset of M, we get the local L^2 and Sobolev spaces $L^2_{\mathrm{loc}}(M, E)$ and $H^1_{\mathrm{loc}}(M, E)$. Now the Sobolev space is independent of the preceding choices.

3. Adapted operators to Dirac-type operators

From now on, we focus on a special kind of operators.

3.1. **Dirac-type operators.**

Definition 3.2. We say that $D : C^\infty(M, E) \to C^\infty(M, F)$ is a *Dirac-type operator* if σ_D satisfies the *Clifford relations*,

$$\sigma_D(\xi)^*\sigma_D(\eta) + \sigma_D(\eta)^*\sigma_D(\xi) \ = \ 2\langle \xi, \eta\rangle \cdot \mathrm{id}_{E_x}, \tag{3.1}$$

$$\sigma_D(\xi)\sigma_D(\eta)^* + \sigma_D(\eta)\sigma_D(\xi)^* \ = \ 2\langle \xi, \eta\rangle \cdot \mathrm{id}_{F_x}, \tag{3.2}$$

for all $x \in M$ and $\xi, \eta \in T^*_x M$.

This definition generalizes the classical Dirac operators on Clifford modules. By (2.1), if D is a Dirac-type operator, so is D^*.

By (3.1) and (3.2), a Dirac-type operator D is elliptic with

$$\sigma_D(\xi)^{-1} \ = \ \sigma_D(\xi)^* \tag{3.3}$$

for unit covector ξ. If D is also formally self-adjoint on E, then (3.1), (3.2) combined with (2.1) gives that

$$\sigma_D(\xi)\sigma_D(\eta) + \sigma_D(\eta)\sigma_D(\xi) \ = \ -2\langle \xi, \eta\rangle \cdot \mathrm{id}_{E_x}.$$

3.3. **Adapted operators on the boundary.** Note that for $x \in \partial M$, one can identify $T^*_x\partial M$ with the space $\{\xi \in T^*_x M : \langle \xi, \tau(x)\rangle = 0\}$. Suppose D is a Dirac-type operator, by (3.1) and (3.3),

$$\sigma_D(\tau(x))^{-1} \circ \sigma_D(\xi) : E_x \to E_x \tag{3.4}$$

is skew-Hermitian, for all $x \in \partial M$ and $\xi \in T^*_x \partial M$.

Definition 3.4. A formally self-adjoint first-order differential operator $A : C^\infty(\partial M, E) \to C^\infty(M, E)$ is called an *adapted operator* to D if the principal symbol of A is given by

$$\sigma_A(\xi) \ = \ \sigma_D(\tau(x))^{-1} \circ \sigma_D(\xi). \tag{3.5}$$

Remark 3.5. Adapted operators always exist and are also Dirac-type. They are unique up to addition of a Hermitian bundle map of E.

Similarly, by (2.1), (3.3) and (3.5) for an adapted operator $\tilde{A}$ to D^*,

$$\begin{aligned}\sigma_{\tilde{A}}(\xi) &= \sigma_{D^*}(\tau(x))^{-1} \circ \sigma_{D^*}(\xi) \\ &= (-\sigma_D(\tau(x))^*)^{-1} \circ (-\sigma_D(\xi)^*) \\ &= \sigma_D(\tau(x)) \circ \sigma_D(\xi)^*.\end{aligned}$$

Again by (3.5),

$$\begin{aligned}\sigma_{\tilde{A}}(\xi) &= \sigma_D(\tau(x)) \circ (\sigma_D(\tau(x)) \circ \sigma_A(\xi))^* \\ &= \sigma_D(\tau(x)) \circ \sigma_A(\xi)^* \circ \sigma_D(\tau(x))^* \\ &= \sigma_D(\tau(x)) \circ \sigma_{-A}(\xi) \circ \sigma_D(\tau(x))^{-1}.\end{aligned}$$

This means that, if A is adapted to D, then

$$\tilde{A} = \sigma_D(\tau) \circ (-A) \circ \sigma_D(\tau)^{-1} \tag{3.6}$$

is an adapted operator to D^*. This is a natural choice of $\tilde{A}$ from A. Moreover, if $E = F$ and D is formally self-adjoint, we can further require that $A = \tilde{A}$, namely, we can find an adapted operator A to D such that

$$A \circ \sigma_D(\tau) = -\sigma_D(\tau) \circ A. \tag{3.7}$$

4. Boundary value problems

In this section, we study the boundary value problems for Dirac-type operators $D : C^\infty(M, E) \to C^\infty(M, F)$ with associated adapted operators $A : C^\infty(\partial M, E) \to C^\infty(\partial M, E)$. Roughly speaking, the boundary conditions are defined to be closed subspaces of a hybrid Sobolev space induced by A. We follow the notations from last section.

4.1. **Sobolev spaces on the boundary.** From Subsection 3.3, we know that an adapted operator A to a Dirac-type operator D is an essentially self-adjoint elliptic operator on the closed manifold ∂M. For any $s \in \mathbb{R}$, the positive operator $(\mathrm{id} + A^2)^{s/2}$ is defined by functional calculus.

Definition 4.2. For any $s \in \mathbb{R}$, the H^s*-norm* on $C^\infty(\partial M, E)$ is defined by

$$\|u\|^2_{H^s(\partial M)} := \|(\mathrm{id} + A^2)^{s/2} u\|^2_{L^2(\partial M)}. \tag{4.1}$$

The Sobolev space $H^s(\partial M, E)$ is the completion of $C^\infty(\partial M, E)$ with respect to this norm. In particular, $H^0(\partial M, E) = L^2(\partial M, E)$.

There is an alternate way to describe $H^s(\partial M, E)$. By spectral theory, A has discrete spectrum consisting of real eigenvalues $\{\lambda_j\}_{j \in \mathbb{Z}}$, each of which has finite multiplicity, so the corresponding eigenspaces V_j are finite-dimensional. Hence we have decomposition of $L^2(\partial M, E)$ into eigenspaces of A:

$$L^2(\partial M, E) = \bigoplus_{\lambda_j \in \mathrm{spec}(A)} V_j. \tag{4.2}$$

In other words, $L^2(\partial M, E)$ has an orthonormal basis $\{u_{j_k}\}_{j \in \mathbb{Z},\, k = 1,2,\ldots,\mathrm{mult}(\lambda_j)}$, of eigensections of A, where for each j, $\{u_{j_k}\}_k$ is a basis of V_j. In terms of such an orthonormal basis, for

$u = \sum_{j,k} a_{j_k} u_{j_k}$ and $s \in \mathbb{R}$, the H^s-norm becomes

$$\|u\|^2_{H^s(\partial M)} \ = \ \sum_{j,k} |a_{j_k}|^2 (1+\lambda_j^2)^s. \tag{4.3}$$

So $H^s(\partial M, E)$ can also be defined to be the subspace of $L^2(\partial M, E)$ such that the right hand side of (4.3) is finite.

Note that there is a perfect pairing between $H^s(\partial M, E)$ and $H^{-s}(\partial M, E)$ for all $s \in \mathbb{R}$. Since ∂M is compact, the *Rellich embedding theorem* asserts that for $s_1 > s_2$, the embedding

$$H^{s_1}(\partial M, E) \ \hookrightarrow \ H^{s_2}(\partial M, E) \tag{4.4}$$

is compact.

For $I \subset \mathbb{R}$, let

$$P_I : L^2(\partial M, E) \ \to \ \bigoplus_{\lambda_j \in I} V_j \tag{4.5}$$

be the orthogonal spectral projection. So

$$P_I : \sum_{j,k} a_{j_k} u_{j_k} \ \mapsto \ \sum_{\lambda_j \in I} a_{j_k} u_{j_k}$$

It's easy to see that

$$P_I(H^s(\partial M, E)) \ \subset \ H^s(\partial M, E)$$

for all $s \in \mathbb{R}$. Then by

$$C^\infty(\partial M, E) \ = \ \bigcap_{s \in \mathbb{R}} H^s(\partial M, E),$$

we have

$$P_I(C^\infty(\partial M, E)) \ \subset \ C^\infty(\partial M, E).$$

Set $H^s_I(A) := P_I(H^s(\partial M, E))$. For $a \in \mathbb{R}$, define the hybrid Sobolev spaces

$$\check{H}(A) \ := \ H^{1/2}_{(-\infty,a)}(A) \ \oplus \ H^{-1/2}_{[a,\infty)}(A), \tag{4.6}$$

$$\hat{H}(A) \ := \ H^{-1/2}_{(-\infty,a)}(A) \ \oplus \ H^{1/2}_{[a,\infty)}(A). \tag{4.7}$$

The corresponding $\check{H}$-, $\hat{H}$-norms are

$$\|u\|^2_{\check{H}(A)} \ := \ \|P_{(-\infty,a)} u\|^2_{H^{1/2}(\partial M)} \ + \ \|P_{[a,\infty)} u\|^2_{H^{-1/2}(\partial M)},$$
$$\|u\|^2_{\hat{H}(A)} \ := \ \|P_{(-\infty,a)} u\|^2_{H^{-1/2}(\partial M)} \ + \ \|P_{[a,\infty)} u\|^2_{H^{1/2}(\partial M)}.$$

The spaces $\check{H}(A)$ and $\hat{H}(A)$ are independent of the choice of a. Since for a different a, the $\check{H}$-, $\hat{H}$-norms only differ by a norm on a finite-dimensional space, thus are equivalent to the original ones. In particular,

$$\hat{H}(A) \ = \ \check{H}(-A). \tag{4.8}$$

Again, there is a perfect pairing between $\check{H}(A)$ and $\hat{H}(A)$.

4.3. **The maximal domain.** Before discussing the properties of $\operatorname{dom} D_{\max}$, we first define a subspace of it as

$$H^1_D(M,E) \;:=\; \operatorname{dom} D_{\max} \cap H^1_{\mathrm{loc}}(M,E).$$

Note that the only difference between $H^1_D(M,E)$ and $\operatorname{dom} D_{\max}$ is the H^1-regularity near the boundary.

The next theorem summarizes the properties of $\operatorname{dom} D_{\max}$ (cf. [4], [5]).

Theorem 4.4. *For a Dirac-type operator $D : C^\infty(M,E) \to C^\infty(M,F)$ with adapted operator $A : C^\infty(\partial M,E) \to C^\infty(\partial M,E)$, we have*

(i) *$C^\infty_c(M,E)$ is dense in $\operatorname{dom} D_{\max}$;*
(ii) *the trace map $Tu := u|_{\partial M}$ on $C^\infty_c(M,E)$ extends uniquely to a surjective bounded linear map $T : \operatorname{dom} D_{\max} \to \check{H}(A)$;*
(iii) $\operatorname{dom} D_{\min} = \{u \in \operatorname{dom} D_{\max} : Tu = 0\}$. *In particular, T induces an isomorphism*

$$\check{H}(A) \;\cong\; \operatorname{dom} D_{\max}/\operatorname{dom} D_{\min};$$

(iv) *for any closed subspace $B \subset \check{H}(A)$, the operator $D_{B,\max}$ with domain*

$$\operatorname{dom} D_{B,\max} \;=\; \{u \in \operatorname{dom} D_{\max} : Tu \in B\}$$

is a closed extension of D between $D_{\min}$ and $D_{\max}$, and any closed extension of D between $D_{\min}$ and $D_{\max}$ is of this form;
(v) *for all $u \in \operatorname{dom} D_{\max}$ and $v \in \operatorname{dom} D^*_{\max}$,*

$$\int_M \langle Du, v\rangle dV \;=\; \int_M \langle u, D^*v\rangle dV \;-\; \int_{\partial M} \langle \sigma_D(\tau)Tu, Tv\rangle dS;$$

(vi) $H^1_D(M,E) = \{u \in \operatorname{dom} D_{\max} : Tu \in H^{1/2}(\partial M,E)\}$.

Remark 4.5. (1) To help understand (ii) and (vi), we recall the *trace theorem* which says that $T : C^\infty_c(M,E) \to C^\infty(\partial M,E)$ extends to a bounded linear map

$$T : H^k_{\mathrm{loc}}(M,E) \;\to\; H^{k-1/2}(\partial M,E)$$

for all $k \geq 1$.

(2) As a consequence of (iii), the topology of $\check{H}(A)$ does not depend on the choice of adapted operator A.

(3) (v) generalizes the Green's formula. The pairing in the last integral is well-defined because $\sigma_D(\tau)$ maps $\check{H}(A)$ to $\hat{H}(\tilde{A})$ by (3.6).

Similar to the elliptic regularity on manifolds without boundary, we have the following *boundary regularity.*

Theorem 4.6. *For any integer $k \geq 0$,*

$$\begin{aligned}&\operatorname{dom} D_{\max} \cap H^{k+1}_{\mathrm{loc}}(M,E)\\ &\quad = \{u \in \operatorname{dom} D_{\max} : Du \in H^k_{\mathrm{loc}}(M,F) \text{ and } P_{[0,\infty)}(Tu) \in H^{k+1/2}(\partial M,E)\}.\end{aligned}$$

In particular,

$$u \in \operatorname{dom} D_{max} \cap H^1_{\mathrm{loc}}(M,E) \iff P_{[0,\infty)}(Tu) \in H^{1/2}(\partial M,E).$$

Note that $P_{[0,\infty)}(Tu) \in H^{1/2}(\partial M,E)$ if and only if $Tu \in H^{1/2}(\partial M,E)$ by (4.6) and Theorem 4.4.(ii).

4.7. **Boundary conditions.** In view of Theorem 4.4.(iv), the definition for boundary conditions becomes natural.

Definition 4.8. A *boundary condition* for D is a closed subspace of $\check{H}(A)$. Following the notation in Theorem 4.4.(iv), we write $D_{B,\max}$ for the operator with boundary condition B.

Regrading $D_{B,\max}$ as an unbounded operator on $L^2(M,E)$, it has an adjoint operator which is also given by a boundary condition.

Theorem 4.9. *Assume that $B \subset \check{H}(A)$ is a boundary condition for D. Let $\tilde{A}$ be adapted to D^*. Then*

$$B^{\mathrm{ad}} \ := \ \{v \in \check{H}(\tilde{A}) : (\sigma_D(\tau)u, v) = 0, \ for\ all\ u \in B\}$$

is a closed subspace of $\check{H}(\tilde{A})$, thus, it is a boundary condition for D^*, where $(\sigma_D(\tau)u, v)$ stands for the L^2-inner product on ∂M. Moreover, $D^*_{B^{\mathrm{ad}},\max}$ is the adjoint operator of $D_{B,\max}$.

Remark 4.10. This theorem basically follows from Theorem 4.4.(v).

Taking local H^1-regularity into consideration, we define

$$\operatorname{dom} D_B \ := \ \{u \in H^1_D(M,E) : Tu \in B\} \ \subset \operatorname{dom} D_{B,\max}.$$

4.11. **Elliptic boundary conditions.** Notice that for a boundary condition B for D, in general, $\operatorname{dom} D_{B,\max} \neq \operatorname{dom} D_B$. That means, the sections in $\operatorname{dom} D_{B,\max}$ are usually not locally H^1-regular. In order to have similar properties as elliptic operators on manifolds without boundary, we need further restriction on the boundary conditions such that the two domains are equal.

Definition 4.12. A boundary condition B is said to be *elliptic* if $B \subset H^{1/2}(\partial M, E)$ and $B^{\mathrm{ad}} \subset H^{1/2}(\partial M, F)$.

Remark 4.13. (1) By Theorem 4.4.(vi), $\operatorname{dom} D_{B,\max} = \operatorname{dom} D_B$ if and only if $B \subset H^{1/2}(\partial M, E)$. So for elliptic boundary condition B, we will not distinguish between these two subspaces, and can regard it as a closed subspace of $\operatorname{dom} D_{\max}$ with respect to the graph norm.

(2) There is an equivalent way to define elliptic boundary conditions, see [4, Definition 7.5] or [5, Definition 3.7]. From there, we also get that B is an elliptic boundary condition if and only if B^{ad} is.

4.14. **The Atiyah-Patodi-Singer boundary condition.** A typical example of elliptic boundary condition is introduced in [3], which is called Atiyah-Patodi-Singer boundary condition, or APS boundary condition for short.

Let $D : C^\infty(M,E) \to C^\infty(M,F)$ be a Dirac-type operator. Assume product structure near the boundary ∂M, such that

$$D \ = \ \sigma_D(\tau)\big(\partial_t + A\big) \tag{4.9}$$

in a tubular neighborhood of ∂M, where t is the normal direction and A is an adapted operator to D. Then by (2.1), (3.3) and the fact that A is formally self-adjoint,

$$\begin{aligned} D^* \ &= \ \big(-\partial_t + A^*\big)\sigma_D(\tau)^* \\ &= \ \sigma_{D^*}(\tau)\big(\partial_t - \sigma_D(\tau)\circ A \circ \sigma_D(\tau)^*\big) \\ &= \ \sigma_{D^*}(\tau)\big(\partial_t + \sigma_D(\tau)\circ(-A)\circ\sigma_D(\tau)^{-1}\big) \\ &= \ \sigma_{D^*}(\tau)\big(\partial_t + \tilde{A}\big), \end{aligned}$$

where $\tilde{A}$ is as in (3.6).

By the spectral theory discussed in Subsection 4.1, we have the decomposition (4.2) of $L^2(\partial M, E)$ and spectral projection P_I as in (4.5). Let $I = (-\infty, 0)$, getting $H^{1/2}_{(-\infty,0)}(A) = P_{(-\infty,0)}(H^{1/2}(\partial M, E))$. We define the *Atiyah-Patodi-Singer boundary condition*

$$B_{\mathrm{APS}} \; := \; H^{1/2}_{(-\infty,0)}(A). \tag{4.10}$$

This is a closed subspace of $\check{H}(A)$ given in (4.6). So APS boundary condition is indeed a boundary condition as defined in Definition 4.8.

By Theorem 4.9, the adjoint boundary condition

$$\begin{aligned} B^{\mathrm{ad}}_{\mathrm{APS}} \;&=\; \{v \in \check{H}(\tilde{A}) : (\sigma_D(\tau)u, v) = 0, \; for\ all\ u \in B_{\mathrm{APS}}\} \\ &=\; \{v \in \check{H}(\tilde{A}) : (u, \sigma_D(\tau)^* v) = 0, \; for\ all\ u \in B_{\mathrm{APS}}\}. \end{aligned}$$

So

$$\sigma_D(\tau)^* B^{\mathrm{ad}}_{\mathrm{APS}} \subset (B_{\mathrm{APS}})^\perp \subset P_{[0,\infty)}(L^2(\partial M, E)).$$

But on the other hand,

$$\sigma_D(\tau)^* B^{\mathrm{ad}}_{\mathrm{APS}} = \sigma_D(\tau)^{-1} B^{\mathrm{ad}}_{\mathrm{APS}} \subset \sigma_D(\tau)^{-1} \check{H}(\tilde{A}) = \check{H}(-A) = \hat{H}(A)$$

by (3.6) and (4.8). Hence

$$\sigma_D(\tau)^* B^{\mathrm{ad}}_{\mathrm{APS}} \subset P_{[0,\infty)}(L^2(\partial M, E)) \cap \hat{H}(A) = H^{1/2}_{[0,\infty)}(A)$$

by (4.7). This gives that

$$B^{\mathrm{ad}}_{\mathrm{APS}} \subset (\sigma_D(\tau)^*)^{-1} H^{1/2}_{[0,\infty)}(A) = \sigma_D(\tau) H^{1/2}_{[0,\infty)}(A) = H^{1/2}_{[0,\infty)}(-\tilde{A}) = H^{1/2}_{(-\infty,0]}(\tilde{A}).$$

On the other hand, one easily verifies that $H^{1/2}_{(-\infty,0]}(\tilde{A}) \subset B^{\mathrm{ad}}_{\mathrm{APS}}$. Therefore the adjoint condition for APS boundary condition is

$$B^{\mathrm{ad}}_{\mathrm{APS}} \;=\; H^{1/2}_{(-\infty,0]}(\tilde{A}).$$

In particular, this means that

Proposition 4.15. *The APS boundary condition defined in* (4.10) *is an elliptic boundary condition.*

5. Fredholm property and relative index theorem

In this section, we consider the operator $D_{B,\max}$ for an elliptic boundary condition B and develop some basic index theory.

5.1. **Invertibility at infinity.** We know that if the manifold M is noncompact without boundary, in general, an elliptic operator on it is not Fredholm. Similarly, for noncompact manifold M with compact boundary ∂M, elliptic boundary condition does not guarantee that the operator is Fredholm. In order to make it, we need proper behavior of the operator at infinity.

Definition 5.2. We say that an operator D is *invertible at infinity* (or *coercive at infinity*) if there exist a constant $C > 0$ and a compact subset $K \Subset M$ such that

$$\|Du\|_{L^2(M)} \;\geq\; C\|u\|_{L^2(M)}, \tag{5.1}$$

for all $u \in C^\infty_c(M, E)$ with $\mathrm{supp}(u) \cap K = \emptyset$.

Remark 5.3. (1) By definition, if M is compact, then D is invertible at infinity.

(2) Boundary conditions have nothing to do with invertibility at infinity since the compact set K can always be chosen such that a neighborhood of ∂M is contained in K.

An important class of examples for operators which are invertible at infinity is the so-called *Callias-type operators* that will be discussed in next section.

5.4. **Fredholmness.** Recall for $\partial M = \emptyset$, a first order essentially self-adjoint elliptic operator which is invertible at infinity is Fredholm (cf. [1, Theorem 2.1]). For $\partial M \neq \emptyset$, we have the following analogous result.

Theorem 5.5. *Assume that $D_{B,\max} : \operatorname{dom} D_{B,\max} \to L^2(M, F)$ is a Dirac-type operator with elliptic boundary condition. If D is invertible at infinity, then $D_{B,\max}$ has finite-dimensional kernel and closed range.*

Before proving it, we first recall a classical result (cf. [12, Proposition 19.1.3]).

Lemma 5.6. *Let X and Y be Banach spaces and $L : X \to Y$ be a bounded linear map. Then the following are equivalent:*

- (i) *Every bounded sequence $\{x_j\}$ in X such that $\{Lx_j\}$ converges in Y has a convergent subsequence in X.*
- (ii) *The operator L has finite-dimensional kernel and closed range.*

Proof of Theorem 5.5. We apply the above lemma. Let $\{u_j\}$ be a bounded sequence in $\operatorname{dom} D_{B,\max}$ such that $Du_j \to v \in L^2(M, F)$. We need to show that $\{u_j\}$ has a convergent subsequence in $\operatorname{dom} D_{B,\max}$.

Assume that D is invertible at infinity as in Definition 5.2. Let $\chi : M \to [0, 1]$ be a cut-off function such that $\chi \equiv 1$ on K and $K' := \operatorname{supp}(\chi)$ is compact. Since B is an elliptic boundary condition, each element in $\operatorname{dom} D_{B,\max}$ is locally H^1. By (5.1) and the fact that D is of Dirac-type, passing to a subsequence if necessary,

$$\begin{aligned}
\|u_j - u_k\|_{L^2(M)} &\leq \|\chi(u_j - u_k)\|_{L^2(M)} + \|(1 - \chi)(u_j - u_k)\|_{L^2(M)} \\
&\leq \|u_j - u_k\|_{L^2(K')} + C^{-1}\|D((1 - \chi)(u_j - u_k))\|_{L^2(M)} \\
&\leq \|u_j - u_k\|_{L^2(K')} + C^{-1}\| - \sigma_D(d\chi)(u_j - u_k)\|_{L^2(M)} \\
&\qquad + C^{-1}\|(1 - \chi)(Du_j - Du_k)\|_{L^2(M)} \\
&\leq C'\|u_j - u_k\|_{L^2(K')} + C^{-1}\|Du_j - Du_k\|_{L^2(M)} \\
&\leq \varepsilon_1 + \varepsilon_2 \to 0,
\end{aligned}$$

where the presence of ε_1 is because of Rellich embedding theorem (4.4), and the presence of ε_2 is by hypothesis. So $\{u_j\}$ is a Cauchy sequence in $L^2(M, E)$ and thus converges in $L^2(M, E)$.

Notice that $\operatorname{dom} D_{B,\max}$ is equipped with graph norm. Now $\{u_j\}$ and $\{Du_j\}$ converge in $L^2(M, E)$ and $L^2(M, F)$, respectively, that is, $\{u_j\}$ converges in the graph norm of D. Hence it converges in $\operatorname{dom} D_{B,\max}$, and $D_{B,\max}$ has finite-dimensional kernel and closed range by Lemma 5.6. □

As an immediate consequence, we get a criteria for $D_{B,\max}$ to be Fredholm.

Corollary 5.7. *Assume that $D_{B,\max} : \operatorname{dom} D_{B,\max} \to L^2(M, F)$ is a Dirac-type operator with elliptic boundary condition. If D and D^* are invertible at infinity, then $D_{B,\max}$ is a Fredholm operator.*

Under such circumstance, we define the L^2*-index* of D subject to the boundary condition B as the integer

$$L^2 - \operatorname{ind} D_{B,\max} \;:=\; \dim\ker D_{B,\max} - \dim\ker D^*_{B^{\mathrm{ad}},\max} \;\in\; \mathbb{Z}.$$

5.8. **Relative index theorem.** After having a well-defined index for an operator with elliptic boundary condition, the relative index theorem can be achieved now. Here we only give a statement, and for more details, see [4], [5]. To simplify notation, we use D_B for $D_{B,\max}$. This will not cause confusion by (1) of Remark 4.13.

Let M_j, $j = 1, 2$ be two manifolds with compact boundary and $D_{j,B_j} : \operatorname{dom} D_{j,B_j} \to L^2(M_j, F_j)$ be two Dirac-type operators with elliptic boundary conditions. Suppose $M'_j \cup_{\Sigma_j} M''_j$ are partitions of M_j into relatively open submanifolds, where Σ_j are compact hypersurfaces of $\mathring{M}_j$. We assume that Σ_j have tubular neighborhoods which are diffeomorphic and the structures on the neighborhoods are isomorphic.

We cut M_j along Σ_j and glue the pieces together interchanging M''_1 and M''_2. In this way we obtain the manifolds

$$M_3 \;:=\; M'_1 \cup_\Sigma M''_2, \qquad\qquad M_4 \;:=\; M'_2 \cup_\Sigma M''_1,$$

where $\Sigma \cong \Sigma_1 \cong \Sigma_2$. Then we get operators D_{3,B_3} and D_{4,B_4} on M_3 and M_4, respectively. The relative index theorem says that

Theorem 5.9. *If D_{j,B_j}, $j = 1, 2, 3, 4$ are all Fredholm operators, then*

$$(L^2 - \operatorname{ind} D_{1,B_1}) + (L^2 - \operatorname{ind} D_{2,B_2}) \;=\; (L^2 - \operatorname{ind} D_{3,B_3}) + (L^2 - \operatorname{ind} D_{4,B_4}).$$

6. Callias-type operators with APS boundary condition

6.1. **Callias-type operators.** Let M be a complete odd-dimensional oriented Riemannian manifold with boundary ∂M. E is a Clifford module over M. Let $D : C_c^\infty(M, E) \to C_c^\infty(M, E)$ be a formally self-adjoint Dirac-type operator, meaning its principal symbol is the Clifford multiplication $c(\cdot)$. Suppose $\Phi \in \operatorname{End}(E)$ is a self-adjoint bundle map (called potential). Then $\mathcal{D} := D + i\Phi$ is again a Dirac-type operator on E with formal adjoint given by

$$\mathcal{D}^* \;=\; D - i\Phi.$$

So

$$\begin{aligned} \mathcal{D}^*\mathcal{D} \;&=\; D^2 + \Phi^2 + i[D, \Phi], \\ \mathcal{D}\mathcal{D}^* \;&=\; D^2 + \Phi^2 - i[D, \Phi], \end{aligned} \tag{6.1}$$

where

$$[D, \Phi] \;:=\; D\Phi \;-\; \Phi D$$

is the commutator of the operators D and Φ.

Definition 6.2. We say that $\mathcal{D}$ is a *Callias-type operator* if

(i) $[D, \Phi]$ is a zeroth order differential operator, i.e. a bundle map;
(ii) there is a compact subset $K \Subset M$ and a constant $c > 0$ such that

$$\Phi^2(x) \;-\; |[D, \Phi](x)| \;\geq\; c$$

for all $x \in M \setminus K$. Here $|[D, \Phi](x)|$ denotes the operator norm of the linear map $[D, \Phi](x) : E_x \to E_x$. In this case, the compact set K is called the *essential support* of $\mathcal{D}$.

Remark 6.3. $\mathcal{D}$ is a Callias-type operator if and only if $\mathcal{D}^*$ is.

The definition implies immediately that

Proposition 6.4. *Callias-type operators are invertible at infinity in the sense of Definition 5.2.*

Proof. Since ∂M is compact, we can always assume that the essential support K contains a neighborhood of ∂M. Thus for all $u \in C_c^\infty(M,E)$ with $\mathrm{supp}(u) \cap K = \emptyset$, $u \in C_{cc}^\infty(M,E)$. Then by Green's formula Proposition 2.2, (6.1), and Definition 6.2,

$$\begin{aligned}\|\mathcal{D}u\|^2_{L^2(M)} &= (\mathcal{D}u,\mathcal{D}u)_{L^2(M)} = (\mathcal{D}^*\mathcal{D}u,u)_{L^2(M)} \\ &= (D^2u,u)_{L^2(M)} + ((\Phi^2 + i[D,\Phi])u,u)_{L^2(M)} \\ &\geq \|Du\|^2_{L^2(M)} + c\|u\|^2_{L^2(M)} \\ &\geq c\|u\|^2_{L^2(M)}.\end{aligned}$$

Therefore $\|\mathcal{D}u\|_{L^2(M)} \geq \sqrt{c}\|u\|_{L^2(M)}$ and $\mathcal{D}$ is invertible at infinity. □

Remark 6.5. When $\partial M = \emptyset$, $\mathcal{D}$ has a unique closed extension to $L^2(M,E)$, and it is a Fredholm operator. Thus one can define its L^2-index,

$$L^2 - \mathrm{ind}\,\mathcal{D} := \dim\{u \in L^2(M,E) : \mathcal{D}u = 0\} - \dim\{u \in L^2(M,E) : \mathcal{D}^*u = 0\}.$$

An exciting result says that this index is equal to the index of a Dirac-type operator on a compact hypersurface outside of the essential support. This was first prove by Callias in [11] for Euclidean space (see also [7]) and was later generalized to manifolds in [2], [13], [10], etc. In [9] and [8], the relationship between such result and cobordism invariance of the index was being discussed for usual and von Neumann algebra cases, respectively.

Remark 6.6. If $\partial M \neq \emptyset$, in general, $\mathcal{D}$ is not Fredholm. By Corollary 5.7, we need an elliptic boundary condition in order to have a well-defined index and study it.

6.7. **The APS boundary condition for Callias-type operators.** We impose the APS boundary condition as discussed in Subsection 4.14 that enables us to define the index for Callias-type operators.

As in Subsection 4.14, we assume the product structure (4.9) for D near ∂M. We also assume that Φ does not depend on t near ∂M. Then near ∂M,

$$\mathcal{D} = c(\tau)\big(\partial_t + A - ic(\tau)\Phi\big) = c(\tau)\big(\partial_t + \mathcal{A}\big),$$

where $\mathcal{A} := A - ic(\tau)\Phi$ is still formally self-adjoint and thus is an adapted operator to $\mathcal{D}$.

Replacing D and A in Subsection 4.14 by $\mathcal{D}$ and $\mathcal{A}$, we define the APS boundary condition B_{APS} as in (4.10) for the Callias-type operator $\mathcal{D}$. It is an elliptic boundary condition. Combining Corollary 5.7, Remark 6.3 and Proposition 6.4, we obtain the Fredholmness for the operator $\mathcal{D}_{B_{\mathrm{APS}},\max}$.

Proposition 6.8. *The operator $\mathcal{D}_{B_{\mathrm{APS}},\max}$ is Fredholm, thus has an index*

$$L^2 - \mathrm{ind}\,\mathcal{D}_{B_{\mathrm{APS}},\max} = \dim\ker\mathcal{D}_{B_{\mathrm{APS}},\max} - \dim\ker\mathcal{D}^*_{B^{\mathrm{ad}}_{\mathrm{APS}},\max} \in \mathbb{Z}.$$

6.9. **The Callias index theorem.** In this subsection, we explain the Callias index theorem mentioned in Remark 6.5 under APS boundary condition.

By definition 6.2, the potential Φ is nonsingular outside of the essential support K. Then over $M \setminus K$, there is a bundle decomposition $E|_{M\setminus K} = E_+ \oplus E_-$, where $E_\pm$ are the positive/negative eigenspaces of Φ. $E_\pm$ are also Clifford modules.

For any subset $U \subset M$, denote $U_\circ := U \setminus \partial M$. Let $L \Subset M$ be any compact subset of M containing a neighborhood of ∂M such that $K_\circ \Subset \mathring{L}$, and $N := \partial L_\circ$ is a compact oriented hypersurface. Thus we can consider Clifford modules $E_{N\pm} := E_\pm|_N$. On $E_{N\pm}$, there is a grading induced by $\nu := ic(\tau)$. Let $\partial_\pm$ be a restriction of D to $E_{N\pm}$ such that ∂ anti-commutes with ν. They are still Dirac-type operators. Now split E_{N+} into $E_{N+}^\pm$ given by

$$E_{N+}^\pm \;=\; \{u \in E_{N+} : \nu u = \pm u\},$$

and denote by $\partial_+^\pm$ the restrictions of ∂_+ to $E_{N+}^\pm$.

After the above setting, the Callias index theorem is stated as

Theorem 6.10. *Let $\mathcal{D} = D + i\Phi : C_c^\infty(M, E) \to C_c^\infty(M, E)$ be a Callias-type operator on an odd-dimensional oriented complete manifold M with compact boundary ∂M. B_{APS} is the APS boundary condition described in Subsection 6.7. Then*

$$L^2 - \operatorname{ind} \mathcal{D}_{B_{\mathrm{APS}},\max} \;=\; \operatorname{ind} \partial_+^+,$$

where $\partial_+^+ : C^\infty(N, E_{N+}^+) \to C^\infty(N, E_{N+}^-)$ is the Dirac-type operator on the compact manifold N without boundary.

The main ingredient in proving this theorem for $\partial M = \emptyset$ (cf. [2]) is relative index theorem. Since we already established such a result in Theorem 5.9, the proof here is essentially the same.

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E-mail address: `shi.pe@husky.neu.edu`