Math 3150 Fall 2015 HW2 Solutions

Problem 1. Let (s_n) be a sequence that converges
(a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.
(b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.
(c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n \in [a, b]$.

Solution.

(a) Let m be the largest integer such that $s_m < a$ and let $s = \lim s_n$. Proceeding by contradiction, suppose that $s < a$. Choose ε such that $0 < \varepsilon < a - s$. Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$s_n < s + \varepsilon < s + a - s = a.$$
In particular, this holds for $n > \max\{N, m\}$, but then $s_n < a$ contradicts maximality of m.

(b) Let m be the smallest integer such that $s_m > b$ and let $s = \lim s_n$. Proceeding by contradiction, suppose that $s > b$. Choose ε such that $0 < \varepsilon < s - b$. Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$b = s - (s - b) < s - \varepsilon < s_n.$$
In particular, this holds for $n > \max\{N, m\}$, but then $s_n > b$ contradicts maximality of m.

(c) By part (a), $s = \lim s_n \geq a$ and by part (b) $s \leq b$, so $s \in [a, b]$.

□

Problem 2. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
(a) Show if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.
(b) Does $\lim x_n$ exist? Explain.
(c) Discuss the apparent contradiction between parts (a) and (b).

Solution.

(a) Suppose $a = \lim x_n$ exists. Then invoking the limit theorem for the identity $x_{n+1} = 3x_n^2$ gives
$$\lim_{n\to\infty} x_{n+1} = 3(\lim_{n\to\infty} x_n)^2 \implies a = 3a^2.$$
The only two solutions to this equation are $a = 0$ or $a = \frac{1}{3}$.

(b) The limit does not exist. In fact, we can show $x_{n+1} \geq 3^n$ for all n (or a lower bound which grows even more quickly if we want). Indeed, $x_2 = 3$, and by induction, $x_{n+1} = 3x_n^2 \geq 3(3^{n-1})^2 = 3^{2n-1} \geq 3^n$. Since 3^n diverges to infinity, it follows that x_n must also.

(c) The application of the limit theorem $\lim(x_n^2) = (\lim x_n)^2$ in part (a) is only valid in case that $\lim x_n$ is a finite real number. □

Problem 3. Assume all $s_n \neq 0$ and the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.
(a) Show that if $L < 1$, then $\lim s_n = 0$.
(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.

Solution.

(a) Define the sequence $r_n = \left|\frac{s_{n+1}}{s_n}\right|$ of positive real numbers, and suppose that $\lim r_n = L < 1$. Choose $a \in \mathbb{R}$ such that $L < a < 1$, and let $\varepsilon = a - L$. Since $r_n \longrightarrow L$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$r_n < L + \varepsilon = a.$$

This implies $|s_{n+1}| < a\,|s_n|$ for all $n \geq N$, and in particular $|s_{N+1}| < a\,|s_N|$. This is the base case for an induction, where $|s_{N+k}| < a^k\,|s_N|$ implies $|s_{N+k+1}| < a\,|s_{N+k}| < a^{k+1}\,|s_N|$, which may be rewritten as the statement $|s_n| < a^{n-N}\,|s_N|$ for all $n > N$. We therefore have

$$0 \leq |s_n| \leq c\,a^n \quad \forall\ n > N,$$

where $c = \frac{|s_N|}{a^N}$ is a constant. Since $a < 1$, the sequence a^n converges to 0, and $c \cdot a^n \to 0$ also. By the squeeze lemma, it follows that $|s_n| \to 0$ which implies $s_n \to 0$.

(b) Define the sequece $t_n = \frac{1}{|s_n|}$. Then supposing that $\lim \left|\frac{s_{n+1}}{s_n}\right| = L > 1$, it follows that $\lim \left|\frac{t_{n+1}}{t_n}\right| = L^{-1} < 1$. By part (a), $\lim t_n = 0$, and by Theorem 9.10, it follows that $\lim |s_n| = +\infty$. □

Problem 4.

(a) Let (s_n) be a sequence in $\mathbb{R}$ such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Solution.

(a) Let $n, k \in \mathbb{N}$. Consider $|s_{n+k} - s_n|$. Adding and subtracting s_{n+k-1}, s_{n+k-2}, ..., s_{n+1} and employing the triangle inequality, we have

$$\begin{aligned}|s_{n+k} - s_n| &\leq |s_{n+k} - s_{n+k-1}| + |s_{n+k-1} - s_{n+k-2}| + \cdots + |s_{n+1} - s_n| \\ &< 2^{-n} + 2^{-(n+1)} + \cdots + 2^{-(n+k-1)}.\end{aligned}$$

Using the identity $1 + r + \cdots + r^l = \frac{1+r^{l+1}}{1-r}$ for $r < 1$ in the case $r = \frac{1}{2}$, $l = k-1$, we have

$$2^{-n} + \cdots + 2^{-(n+k-1)} = 2^{-n}\frac{1 + 2^{-k}}{\frac{1}{2}} < 2^{-n}\frac{1}{\frac{1}{2}} = 2^{-n+1},$$

thus

$$|s_{n+k} - s_n| < 2^{-n+1}. \tag{1}$$

To prove that s_n is Cauchy, given $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $2^{-N+1} < \varepsilon$. (This is possible since $2^{-n+1} \to 0$ as $n \to \infty$.) Then for any pair $m, n \geq N$, (without loss of generality, $m \geq n$ so $m = n + k$ for some $k \geq 0$),

$$|s_m - s_n| = |s_{n+k} - s_n| < 2^{-n+1} \leq 2^{-N+1} < \varepsilon.$$

Since (s_n) is Cauchy and $\mathbb{R}$ is complete, we conclude that (s_n) converges.

(b) The result is false if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$. As a counter-example, let $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ (the partial summations of the harmonic series). Then $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$, but the sequence (s_n) diverges to infinity. (One way to see this is as follows:

$$\begin{aligned}s_{2^k} &= 1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \cdots + \tfrac{1}{8}) + (\tfrac{1}{9} + \cdots + \tfrac{1}{16}) + \cdots + (\tfrac{1}{2^{k-1}+1} + \cdots + \tfrac{1}{2^k}) \\ &\geq 1 + (\tfrac{1}{2}) + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \cdots + \tfrac{1}{8}) + (\tfrac{1}{16} + \cdots + \tfrac{1}{16}) + \cdots + (\tfrac{1}{2^k} + \cdots + \tfrac{1}{2^k}) \\ &= 1 + (\tfrac{1}{2}) + (\tfrac{2}{4}) + (\tfrac{4}{8}) + (\tfrac{8}{16}) + \cdots + \tfrac{2^{k-1}}{2^k} = \frac{k+2}{2}.\end{aligned}$$

Given any $M > 0$ we can choose a k such that $\frac{k+2}{2} > M$, and so $s_n > M$ for $n = 2^k$; hence $s_n \to +\infty$.) □

Problem 5. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.
(c) Show (s_n) is a decreasing sequence.
(d) Show $\lim s_n$ exists and find $\lim s_n$.

Solution.
(a) $s_2 = \frac{2}{3}$, $s_3 = \frac{5}{9}$, $s_4 = \frac{14}{27}$.
(b) $s_1 = 1 > \frac{1}{2}$ holds. By induction, supposing that $s_n > \frac{1}{2}$, we have

$$s_{n+1} = \tfrac{1}{3}(s_n + 1) > \tfrac{1}{3}(\tfrac{1}{2} + 1) = \tfrac{1}{2},$$

so $s_n > \frac{1}{2}$ for all n.
(c) Let $r_n = s_n - s_{n+1}$. We will show by induction that $r_n \geq 0$ for all n. We have $r_1 = 1 - \frac{2}{3} = \frac{1}{3} > 0$. Assuming $r_n \geq 0$,

$$r_{n+1} = s_n - s_{n+1} = \tfrac{1}{3}\big((s_{n-1} + 1) - (s_n + 1)\big) = \tfrac{1}{3}\big(s_{n-1} - s_n\big) = \tfrac{1}{3}r_n \geq 0,$$

completing the inductive step. Thus (s_n) is decreasing.
Alternatively, (not using induction),

$$\begin{aligned} s_n &> \tfrac{1}{2} \\ \implies \tfrac{2}{3}s_n &> \tfrac{1}{3} \\ \implies \tfrac{1}{3}(s_n + 1) &< s_n \\ \implies s_{n+1} &< s_n, \end{aligned}$$

which holds for all n by the previous part.
(d) Since (s_n) is a decreasing sequence which is bounded below, it converges to some $s = \lim s_n$. Using the limit theorem,

$$\begin{aligned} \lim s_{n+1} &= \tfrac{1}{3}(\lim s_n + 1) \\ \implies s &= \tfrac{1}{3}(s + 1) \\ \implies s &= \tfrac{1}{2}. \quad \square \end{aligned}$$

Problem 6. Let (s_n) be the sequence of numbers in Fig. 11.2 in the book.
(a) Find the set S of subsequential limits of (s_n).
(b) Determine $\limsup s_n$ and $\liminf s_n$.

Solution.
(a) We claim that $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Indeed, for any $\frac{1}{n}$, there are infinitely many $k \in \mathbb{N}$ such that $s_k = \frac{1}{n}$, which implies that (s_n) has a constant subsequence $(\frac{1}{n}, \frac{1}{n}, \ldots)$. In the case of 0, for any $\varepsilon > 0$, there are infinitely many s_k such that $|s_k - 0| < \varepsilon$; indeed, we may take n such that $\frac{1}{n} < \varepsilon$ and consider the constant subsequence $(\frac{1}{n}, \frac{1}{n}, \ldots)$ again. There are no other subsequential limits.
(b) $\liminf s_n = \inf S = 0$ and $\limsup s_n = \sup S = 1$. $\square$