MATH 3150 — HOMEWORK 5

Problem 1 (p. 172, #1)**.** Which of the following sets are connected? Which are compact?
(a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 1\}$
(b) $\{x \in \mathbb{R}^n \mid \|x\| \leq 10\}$
(c) $\{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\}$
(d) $\mathbb{Z} = \{\text{integers in } \mathbb{R}\}$
(e) a finite set in $\mathbb{R}$
(f) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (Be careful with the case $n = 1$!)
(g) Boundary of the unit square in $\mathbb{R}^2$
(h) The boundary of a bounded set in $\mathbb{R}$
(i) The rationals in $[0, 1]$
(j) A closed set in $[0, 1]$

Problem 2 (p. 191, #4)**.** Let $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous, $x, y \in A$ and $c : [0, 1] \longrightarrow A \subset \mathbb{R}^n$ be a continuous curve joining x and y. Show that along this curve, f attains its maximum and minimum values (among all values along the curve).

Problem 3 (p. 193, #3)**.** Let $f : [0, 1] \longrightarrow [0, 1]$ be continuous. Prove that f has a fixed point (i.e. a point $x \in [0, 1]$ such that $f(x) = x$).

Problem 4 (p. 174, #21)**.**

(a) Prove that a set $A \subset (M, d)$ is connected if and only if $\emptyset$ and A are the only subsets of A that are open and closed relative to A. (A set $U \subset A$ is called *open relative to* A if $U = V \cap A$ for some open set $V \subset M$; 'closed relative to A' is defined similarly.)
(b) Prove that $\emptyset$ and $\mathbb{R}^n$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.

Problem 5. Let (M_1, d_1) and (M_2, d_2) be metric spaces with compact sets $K_1 \subset M_1$ and $K_2 \subset M_2$. Show that $K_1 \times K_2$ is a compact subset of the space $(M_1 \times M_2, d = d_1 + d_2)$. (The metric d on the product $M_1 \times M_2$ is defined by $d\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$.)