MATH 3150 — HOMEWORK 4

Problem 1 (p. 125, #2)**.** Let (M, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that M is complete.

Problem 2 (p. 231, #1)**.**

(a) Prove directly (i.e. with 'ε's and 'δ's) that the function $1/x^2$ is continuous on $(0, \infty)$.
(b) A constant function $f : A \longrightarrow \mathbb{R}^m$ is a function such that $f(x) = f(y)$ for all $x, y \in A$. Show that f is continuous.
(c) Is the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(y) = 1/(y^4 + y^2 + 1)$ continuous? Justify your answer.

Problem 3. Define maps $s : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $m : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as addition and scalar multiplication:

$$s(x, y) = x + y, \quad \text{and} \quad m(\lambda, x) = \lambda x.$$

Show that these maps are continuous.

Problem 4 (p. 182, #5, p. 184, #3)**.**

(a) Give an example of a continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ and an open subset $A \subset \mathbb{R}$ such that $f(A)$ is *not* open.
(b) Give an example of a continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a closed subset $B \subset \mathbb{R}$ such that $f(B)$ is *not* closed.

Problem 5 (p. 232, #9)**.** Prove the following "gluing lemma": *Let $f : [a, b] \longrightarrow \mathbb{R}^m$ and $g : [b, c] \longrightarrow \mathbb{R}^m$ be continuous, and such that $f(b) = g(b)$. Define $h : [a, c] \longrightarrow \mathbb{R}^m$ by $h = f$ on $[a, b]$ and $h = g$ on $[b, c]$. Then h is continuous.* Generalize this result to sets $A, B \subset (M, d)$ in a metric space, with functions $f : A \longrightarrow (N, \rho)$ and $g : B \longrightarrow (N, \rho)$.