THE CHAIN RULE

Theorem (Chain Rule)**.** *Let $f : (a,b) \longrightarrow \mathbb{R}$ be differentiable at $x_0 \in (a,b)$ and let $g : B \longrightarrow \mathbb{R}$ be differentiable at $f(x_0) \in B$, where B contains the range $f((a,b))$ of f. Then $g \circ f : (a,b) \longrightarrow \mathbb{R}$ is differentiable at x_0, with derivative*

$$(g \circ f)'(x_0) = g'\big(f(x_0)\big)\, f'(x_0).$$

Proof. We must show that

$$\lim_{x \longrightarrow x_0} \frac{g\big(f(x)\big) - g\big(f(x_0)\big) - g'\big(f(x_0)\big)\, f'(x_0)(x - x_0)}{x - x_0} = 0.$$

Equivalently, given any $\varepsilon > 0$, we must show that there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$\frac{\big|g\big(f(x)\big) - g\big(f(x_0)\big) - g'\big(f(x_0)\big)\, f'(x_0)(x - x_0)\big|}{|x - x_0|} < \varepsilon.$$

Thus suppose $\varepsilon > 0$ is given. For convenience of notation, write $y = f(x)$ and $y_0 = f(x_0)$. By differentiability of g at $y_0 = f(x_0)$, there exists $\delta_{g'} > 0$ such that

$$|y - y_0| < \delta_{g'} \implies \frac{|g(y) - g(y_0) - g'(y_0)(y - y_0)|}{|y - y_0|} < \frac{\varepsilon}{3\,|f'(x_0)|}.$$

Next—since differentiability implies continuity, so f is continuous at x_0—given this $\delta_{g'}$, there exists a $\delta_f > 0$ such that

$$|x - x_0| < \delta_f \implies |y - y_0| = |f(x) - f(x_0)| < \delta_{g'}.$$

Finally, since f is differentiable at x_0, there exists a $\delta_{f'} > 0$ such that

$$|x - x_0| < \delta_{f'} \implies \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} < \min\left(\frac{\varepsilon}{3\,|g'(y_0)|}, \big|f'(x_0)\big|\right).$$

Putting these all together, we let $\delta = \min(\delta_{g'}, \delta_f, \delta_{f'})$. Then whenever $|x - x_0| < \delta$, we have

$$
\begin{aligned}
&\frac{\bigl|g\bigl(f(x)\bigr) - g\bigl(f(x_0)\bigr) - g'\bigl(f(x_0)\bigr)\, f'(x_0)(x - x_0)\bigr|}{|x - x_0|} \\
&= \frac{|g(y) - g(y_0) - g'(y_0)\, f'(x_0)(x - x_0)|}{|x - x_0|} \\
&= \frac{|g(y) - g(y_0) - g'(y_0)(y - y_0) + g'(y_0)(y - y_0) - g'(y_0)\, f'(x_0)(x - x_0)|}{|x - x_0|} \\
&\le \frac{|g(y) - g(y_0) - g'(y_0)(y - y_0)|}{|x - x_0|} + \frac{|g'(y_0)(y - y_0) - g'(y_0)\, f'(x_0)(x - x_0)|}{|x - x_0|} \\
&< \frac{\varepsilon}{3\,|f'(x_0)|} \frac{|y - y_0|}{|x - x_0|} + \frac{\bigl|g'(y_0)\bigl(f(x) - f(x_0) - f'(x_0)(x - x_0)\bigr)\bigr|}{|x - x_0|} \\
&< \frac{\varepsilon}{3\,|f'(x_0)|} \frac{|f(x) - f(x_0)|}{|x - x_0|} + \frac{|g'(y_0)|\,\varepsilon}{3\,|g'(y_0)|} \\
&= \frac{\varepsilon}{3\,|f'(x_0)|} \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0) + f'(x_0)(x - x_0)|}{|x - x_0|} + \frac{\varepsilon}{3} \\
&\le \frac{\varepsilon}{3\,|f'(x_0)|} \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} + \frac{\varepsilon}{3\,|f'(x_0)|} \frac{|f'(x_0)(x - x_0)|}{|x - x_0|} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3\,|f'(x_0)|} \bigl|f'(x_0)\bigr| + \frac{\varepsilon}{3\,|f'(x_0)|} \bigl|f'(x_0)\bigr| + \frac{\varepsilon}{3} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
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$\square$