MATH 3150 – HOMEWORK 3

Problem 1 (p. 52, #4)**.**

(a) Let x_n be a Cauchy sequence. Suppose that for every $\epsilon > 0$ there is some $n > 1/\epsilon$ such that $|x_n| < \epsilon$. Prove that $x_n \longrightarrow 0$.

(b) Show that the hypothesis that x_n be Cauchy is necessary, by coming up with an example of a sequence x_n which does not converge, but which has the other property: that for every $\epsilon > 0$ there exists some $n > 1/\epsilon$ such that $|x_n| < \epsilon$.

Problem 2 (p. 99 #15)**.** Let x_n be a sequence in $\mathbb{R}$ such that $|x_n - x_{n+1}| \leq \frac{1}{2} |x_{n-1} - x_n|$. Show that x_n is a Cauchy sequence.

Problem 3. Prove that an Archimedean ordered field in which every Cauchy sequence converges is complete (i.e. has the monotone sequence property). Here are some suggested steps:

(a) Denote the field by $\mathbb{F}$, and suppose x_n is a monotone increasing sequence bounded above by some $M \in \mathbb{F}$.

(b) Proceeding by contradiction, suppose x_n is not Cauchy. Deduce the existence of a subsequence $y_k = x_{n_k}$ with the property that

$$y_k \geq y_{k-1} + \epsilon, \ \forall \ k$$

for some fixed positive number $\epsilon > 0$ which does not depend on k.

(c) Using the Archimedean property, argue that y_k cannot be bounded above by M, hence obtaining a contradiction.

(d) Conclude that x_n converges.

Problem 4 (p. 56 #1)**.** Let $x_n = 3 + (-1)^n(1 + 1/n)$. Calculate $\liminf x_n$ and $\limsup x_n$, and justify your answers.

Problem 5 (p. 56 # 3)**.** Let x_n be a sequence in $\mathbb{R}$ with $a = \liminf x_n$ and $b = \limsup x_n$ both finite. Show that x_n has subsequences u_n and l_n such that $u_n \longrightarrow b$ and $l_n \longrightarrow a$.