MATH 3150 – HOMEWORK 10

Problem 1. Prove the following well-known calculus rule, using the definition of the derivative: If $f(x) = x^n$, $n \in \mathbb{N}$, then f is differentiable and $f'(x) = nx^{n-1}$.

Problem 2 (p. 235, #37)**.** Prove the following intermediate value theorem for derivatives: If f is differentiable at all points of $[a, b]$ and if $f'(a)$ and $f'(b)$ are non-zero, with opposite signs, then there is a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$. (Note that we do *not* assume that f' is continuous, just that it exists at each $x \in [a, b]$.)

Problem 3 (p. 235, #38)**.** A real-valued function defined on (a, b) is called *convex* when the following inequality holds for $x, y \in (a, b)$ and $t \in [0, 1]$:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

If f has a continuous second derivative and $f'' > 0$, show that f is convex.

[Hint: Graphically, the convexity condition says that at each point between a given x and y, the graph of f lies on or below the straight line between $(x, f(x))$ and $(y, f(y))$. Suppose by contradiction that the graph of f lies *above* such a straight line at some point. What can you say about the derivative of f in such a region, and what does $f'' > 0$ imply?]

Problem 4 (p. 236, #43)**.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and set $F(x) = \int_0^{x^2} f(y)\,dy$. Prove that $F'(x) = 2xf(x^2)$. State a more general theorem.

Problem 5 (p. 336, #44)**.** Let $f : [0, 1] \longrightarrow \mathbb{R}$ be Riemann integrable and suppose that for every a, b with $0 \leq a < b \leq 1$ there exists a c with $a < c < b$ with $f(c) = 0$. Prove that $\int_0^1 f\,dx = 0$. Must f be zero? What if f is continuous?