MATH 3150 - HOMEWORK 1

Problem 1. Prove the following proposition.

Proposition. In an ordered field, the following properties hold:

- (i) Unique identities. If a + x = a for every a, then x = 0. If $a \cdot x = a$ for every a, then x = 1.
- (ii) Unique inverses. If a + x = 0, then x = -a. If ax = 1, then $x = a^{-1}$.
- (iii) No divisors of zero. If xy = 0, then x = 0 or y = 0.
- (iv) Cancellation for addition. If a + x = b + x then a = b. If $a + x \le b + x$, then $a \le b$.
- (v) Cancellation for multiplication. If ax = bx and $x \neq 0$, then a = b. If $ax \geq bx$ and x > 0, then $a \geq b$.
- (vi) $0 \cdot x = 0$ for every x.
- (vii) -(-x) = x for every x.
- (viii) $-x = (-1) \cdot x$ for every x.
- (ix) If $x \neq 0$, then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.
- (x) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.
- (xi) If $x \le y$ and $0 \le z$, then $xz \le yz$. If $x \le y$ and $z \le 0$, then $yz \le xz$.
- (xii) If $x \le 0$ and $y \le 0$, then $xy \ge 0$. If $x \le 0$ and $0 \le y$, then $xy \le 0$.
- (xiii) 0 < 1.
- (xiv) For any $x, x^2 \ge 0$.

Problem 2. Give an example of a field with only three elements. Prove that it cannot be made into an ordered field.

Problem 3. Show that $3^n/n!$ converges to 0.

Problem 4. Let $x_n = \sqrt{n^2 + 1} - n$. Compute $\lim_{n \to \infty} x_n$.