

18.03 Final Exam Review

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1 Definitions & Terminology

1.1 Single Equations

A general differential equation looks like

$$F(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0$$

The **order** of the equation is *the highest order derivative of $x(t)$ which occurs in the equation*. In the above, the order is n .

A **linear** equation is one which looks like

$$a_0(t)x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \dots + a_0(t)x(t) = f(t)$$

The $a_i(t)$ are called the **coefficients**. We say the equation is **constant coefficient** if $a_i(t) = a_i = \text{constant for all } i$. If $f(t) = 0$, then the equation is **homogeneous**. For constant coefficient linear equations, we sometimes use the *operator notation*

$$p(D)x(t) = (a_0D^n + \dots + a_0)x(t) = a_0x^{(n)}(t) + \dots + a_0x(t) = f(t)$$

where $p(D)$ is the **operator**.

For a linear equation

$$p(D)x(t) = f(t)$$

we often refer to $f(t)$ as the **input**, and the to solution $x(t)$ as the **response**.

An important property of linear equations is that solutions to different inputs *add*. That is,

$$\left. \begin{array}{l} p(D)x_1 = f_1(t) \\ p(D)x_2 = f_2(t) \end{array} \right\} \implies p(D)(ax_1(t) + bx_2(t)) = af_1(t) + bf_2(t)$$

The solution to an n th order equation will have a **general solution** with n parameters. If the equation is linear, the general solution looks like

$$x(t) = x_h(t) + x_p(t) = \underbrace{c_1x_1(t) + \dots + c_nx_n(t)}_{x_h(t)} + x_p(t)$$

where the c_i are the parameters. The factors $x_1(t), \dots, x_n(t)$ are independent solutions to the **associated homogeneous equation** which is when we set $f(t) = 0$.

We call $x_h(t) = c_1x_1(t) + \dots + c_nx_n(t)$ **the homogeneous solution** and $x_p(t)$ **a particular solution**¹

¹Note that it is not quite correct to say "the particular solution" since particular solutions are not unique. $x_p(t)$ can be *any* solution to the equation. Any two particular solutions will differ by an element of the homogeneous solution. However, most of the techniques we use for finding an $x_p(t)$ do give us a preferred one, so there's not too much harm in thinking of the $x_p(t)$ we get as "the" particular solution.

1.2 Systems of Equations

A system of differential equations looks like

$$\begin{aligned}x'_1 &= f_1(t, x_1, \dots, x_n) \\x'_2 &= f_2(t, x_1, \dots, x_n) \\&\dots \\x'_n &= f_n(t, x_1, \dots, x_n)\end{aligned}$$

which can be expressed in **vector notation**

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

A **linear system** is one which can be written

$$\begin{aligned}x'_1 &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \dots + a_{1n}(t)x_n(t) + f_1(t) \\x'_2 &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \dots + a_{2n}(t)x_n(t) + f_2(t) \\&\dots \\x'_n &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \dots + a_{nn}(t)x_n(t) + f_n(t)\end{aligned}$$

which can be expressed using **matrix notation**

$$\mathbf{x}' = A(t)\mathbf{x}(t) + \mathbf{f}(t), \quad A(t) = \begin{bmatrix} a_{11} & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

If $\mathbf{f}(t) = 0$, the system is **homogeneous**. As in the case of single linear equations, the **general solution** to a linear system has the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = \underbrace{c_1\mathbf{x}_1(t) + \dots + c_n\mathbf{x}_n(t)}_{\mathbf{x}_h(t)} + \mathbf{x}_p(t)$$

where $\mathbf{x}_h(t)$ is the **associated homogeneous solution** (where we set $\mathbf{f} = 0$), and $\mathbf{x}_p(t)$ is any **particular solution**. The general homogeneous solution \mathbf{x}_h consists of an arbitrary linear combination of n independent solutions $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$.

1.3 Initial Value Problems

As we know, an n th order equation (or first order system of n equations) has a general solution depending on n parameters. To fix these parameters uniquely, we often require the solution to satisfy an **initial condition**. A differential equation together with an initial condition is known as an **initial value problem**.

For an n th order (linear) equation, a typical initial value problem looks like

$$a_0x^{(n)}(t) + a_1x^{(n-1)}(t) + \dots + a_0x(t) = f(t), \quad x(t_0) = v_1, x'(t_0) = v_2, \dots, x^{(n-1)}(t_0) = v_n$$

where the v_i are some specified constants. Note that for an n th order equation, we need to specify n conditions (on x and its first $n - 1$ derivatives) at $t = t_0$ to get a unique solution. In many cases, $t_0 = 0$.

For a first order system of n equations, a typical initial value problem looks like

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

where \mathbf{x}_0 is a specified constant vector.

2 First Order Equations

2.1 Separable Equations

A differential equation

$$x' = f(t, x)$$

is **separable** if

$$f(t, x) = g(t)h(x)$$

We can solve it by collecting everything involving x on the left and t on the right, to get

$$\frac{x'}{h(x)} = \frac{1}{h(x)} \frac{dx}{dt} = g(t)$$

Now we can integrate both sides

$$\int \frac{1}{h(x)} \frac{dx}{dt} dt = \int \frac{dx}{h(x)} = \int g(t) dt$$

and solve the result for x . The constant of integration serves as the free parameter in the general solution for x .

Example. Solve $2x' = t^2(1 - x^2)$ for $x(t)$ (i.e. find the general solution).

Solution. Write

$$\frac{2x'}{1 - x^2} = t^2$$

and integrate. Use partial fraction decomposition to write

$$\frac{2}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x} = \frac{A(1 - x) + B(1 + x)}{1 - x^2}$$

and so

$$\left. \begin{array}{l} A + B = 2 \\ B - A = 0 \end{array} \right\} \implies B = A = 1$$

We get

$$\begin{aligned} \int \frac{dx}{1 + x} + \int \frac{dx}{1 - x} &= \int t^2 dt \\ \ln(1 + x) - \ln(1 - x) &= \ln\left(\frac{1 + x}{1 - x}\right) = \frac{t^3}{3} + c \\ \left(\frac{1 + x}{1 - x}\right) &= ke^{t^3/3} \\ x &= \frac{ke^{t^3/3} - 1}{ke^{t^3/3} + 1} \end{aligned}$$

□

2.2 Linear Equations

A first order **linear** equation has the general form

$$x' + p(t)x = q(t)$$

We can solve *any* linear equation (even with variable coefficients) by the method of **integrating factors**. We multiply the equation on both sides by a function $u(t)$

$$u(t)x' + u(t)p(t)x = u(t)q(t)$$

and we want to write the left hand side as

$$u(t)x' + u(t)p(t)x = (u(t)x)' = u(t)x' + u'(t)x$$

which will be true if $u' = u(t)p(t)$. Thus we take $u(t) = \exp\left(\int p(t) dt\right)$. Then the equation

$$(u(t)x)' = \left(e^{\int p(t) dt} x\right)' = e^{\int p(t) dt} q(t)$$

is separable and can be integrated. The constant of integration serves as the parameter for the general solution.

Example. Find the solution to $x' + 2x = t$, where $x(0) = 0$.

Solution. The integrating factor $u(t)$ above must satisfy

$$\frac{u'}{u(t)} = 2 \implies u(t) = e^{2t}$$

Thus the equation can be written

$$(e^{2t}x)' = te^{2t}$$

We integrate, using integration by parts

$$\begin{aligned} \int \frac{d}{dt} (e^{2t}x) dt &= \int te^{2t} dt \\ &= t\frac{e^{2t}}{2} - \int \frac{e^{2t}}{2} dt \\ &= \frac{1}{2} \left(te^{2t} - \frac{e^{2t}}{2} \right) + c \end{aligned}$$

So

$$e^{2t}x(t) = \frac{1}{2} \left(te^{2t} - \frac{e^{2t}}{2} \right) + c \implies x(t) = \frac{t}{2} - \frac{1}{4} + ce^{-2t}$$

is the general solution. To satisfy the initial condition, set

$$x(0) = -\frac{1}{4} + c \implies c = \frac{1}{4}$$

so the solution to the initial value problem is

$$x(t) = \frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4}$$

□

When dealing with sines and cosines, it is usually easiest to complexify the problem as illustrated in the following example.

Example. What is the response of $x' + kx = q(t)$ to the input $q(t) = k \sin \omega t$? (Assume $k > 0$, so the system is stable; we're only interested in the steady state, so assume a convenient value for $x(0)$.) Express your answer in the form

$$x(t) = A \sin(\omega t - \phi)$$

A is the **amplitude response**, and ϕ is the **phase shift**.

Solution. It is convenient to consider this as “half” of the complex problem (see appendix A for complex variables techniques)

$$\tilde{x}' + k\tilde{x} = ke^{i\omega t} = k \cos \omega t + ik \sin \omega t$$

where we are interested in the imaginary part of the solution $x = \text{Im}(\tilde{x})$. We solve according to the methods described above, taking the arbitrary constant to be 0 for simplicity (as noted, we are interested in the form of the steady state response, so the value of $\tilde{x}(0)$ is unimportant).

$$\begin{aligned} (e^{kt}\tilde{x})' &= ke^{kt}e^{i\omega t} = ke^{(k+i\omega)t} \\ e^{kt}\tilde{x} &= \frac{ke^{(k+i\omega)t}}{(k+i\omega)} + c \\ \tilde{x} &= \frac{ke^{i\omega t}}{k+i\omega} = \frac{e^{i\omega t}}{1+i\omega/k} \end{aligned}$$

There are a few methods to find the imaginary part of the right hand side. However, because we want to write our solution in the specified form, it is best to do the following. Write $1 + i\omega/k$ in polar form,

$$1 + i\omega/k = \sqrt{1 + \omega^2/k^2} e^{i \tan^{-1}(\omega/k)}$$

Then

$$\frac{e^{i\omega t}}{1 + i\omega/k} = (1 + \omega^2/k^2)^{-1/2} e^{i(\omega t - \phi)}$$

where $\phi = \tan^{-1} \omega/k$. Then the imaginary part (and therefore the solution we seek) is

$$y = \text{Im}(\tilde{y}) = (1 + \omega^2/k^2)^{-1/2} \sin(\omega t - \phi), \quad \phi = \tan^{-1} \omega/k$$

We obtain $A = (1 + \omega^2/k^2)^{-1/2}$ for the amplitude response, and $\phi = \tan^{-1} \omega/k$ for the phase shift.

We note in passing that repeating the same calculation for $q(t) = \cos \omega t$ would be identical, but for taking the *real* part at the end instead of the imaginary part, which gives a similar answer with sin replaced by cos. \square

Often a nonlinear first order equation can be transformed into one which is either linear or separable by an appropriate **change of variables**. The appropriate choice of variables depends on the equation and will be specified in a given problem. **Note:** When applying a change of variables, you must be careful to transform derivatives correctly. Use the chain rule to get a formula for derivatives of the old variable in terms of the new.

Example. Find the general solution to $xy^2y' = x^3 + y^3$, by using the change of variables $y(x) \mapsto z(x) = y/x$.

Solution. First we divide both sides of the equation by xy^2 to get

$$y' = \frac{x^3 + y^3}{xy^2}$$

Then we notice that if we divide the top and bottom of the right hand side by y^3 we get

$$y' = \frac{\frac{x^3}{y^3} + 1}{\frac{x}{y}} = \frac{x^2}{y^2} + \frac{y}{x}$$

Taking $z = y/x$ as suggested, we see that the right hand side can be written as $F(z) = z^{-2} + z$. To write the left hand side in terms of the new variable, we have to use the chain rule:

$$y' = (xz(x))' = xz' + z$$

Plugging in, we get an equation which is separable in z and x :

$$xz' = z^{-2} + z - z = z^{-2}$$

Separate as

$$\begin{aligned} z^2 z' &= \frac{1}{x} \\ \int z^2 dz &= \int \frac{dx}{x} \\ \frac{z^3}{3} &= \ln(x) + c = \ln(x) + \ln(c') = \ln(c'x) \\ z &= (\ln [(c'x)^3])^{1/3} \end{aligned}$$

Now remember to substitute back $y = xz$, so that our (general) solution is

$$y = x (\ln [(c'x)^3])^{1/3}$$

□

2.3 Graphical Methods

2.3.1 Isoclines

We can obtain a graphical picture of the solutions to a first order equation

$$x' = f(x, t)$$

by drawing **isoclines**. A solution to the above equation will be a curve $x(t)$ in the x - t plane, and note that the equation says that the *slope* of such a curve at a point (t_0, x_0) must be equal to $f(x_0, t_0)$. So the method of isoclines works as follows

1. For various choices of c , plot the level curves defined by

$$f(x, t) = c$$

these are the **isoclines**. By the above discussion, any solution curve touching the curve $f(x, t) = c$ must have slope equal to c there. So we draw little hashes with slope equal to c along the corresponding isocline.

2. Sketch solution curves in the plane by demanding that their slope at each isocline match the slope of the hashes on that isocline. Note that solutions cannot cross one another.
3. In particular, if an isocline has hashes which are *tangent to the isocline itself*, then that isocline is also a solution curve, and no other solution may cross it.

Example. Draw isoclines with slope 0, 1, -1, 2 and -2, and sketch some solution curves for the equation

$$x' = -2xt$$

Solution. The plot appears in figure 1. The 0 isocline corresponds to

$$xt = 0$$

whose solutions are both the x and t axis. In particular the t axis itself is a solution $x = 0$. The other curves are hyperbolas.

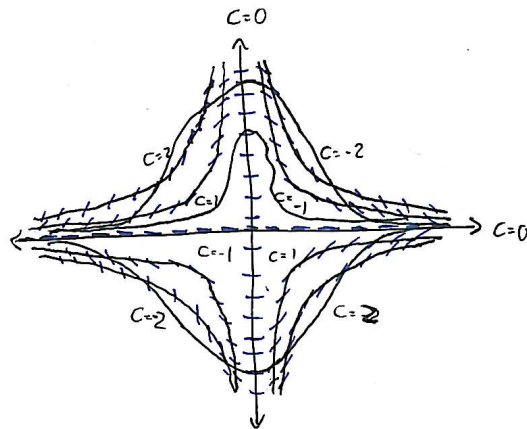


Figure 1: Isoclines

Note that solutions appear to be gaussians, which can be verified by solving this (separable!) equation exactly:

$$\begin{aligned} \int \frac{dx}{x} &= \int -2t dt \\ \ln x &= -t^2 + c \\ x &= c'e^{-t^2} \end{aligned}$$

□

2.3.2 Phase Portraits

If we have an **autonomous** equation, i.e.

$$x' = f(x)$$

where the right hand side doesn't depend on t , we use a slightly different method, known as a **phase portrait**. The isocline picture for an autonomous equation is somewhat redundant, since all of the isoclines are horizontal (constant in t)! So we can “collapse” the picture to be a one dimensional picture consisting of the x -axis only (not the t -axis!). We proceed as follows

1. Plot the **critical points**

$$x_0 \text{ such that } f(x_0) = 0$$

These correspond to *constant* solutions $x(t) = x_0$.

2. In between the critical points, we draw arrows going to the left or right according to whether²

$$\begin{array}{llll} f(x) > 0 & \iff & x(t) \text{ increasing} & \iff & \text{arrow right} \\ f(x) < 0 & \iff & x(t) \text{ decreasing} & \iff & \text{arrow left} \end{array}$$

²It is useful to plot $f(x)$ on a vertical axis over the x -axis to make it obvious where the critical points are and where f is positive/negative.

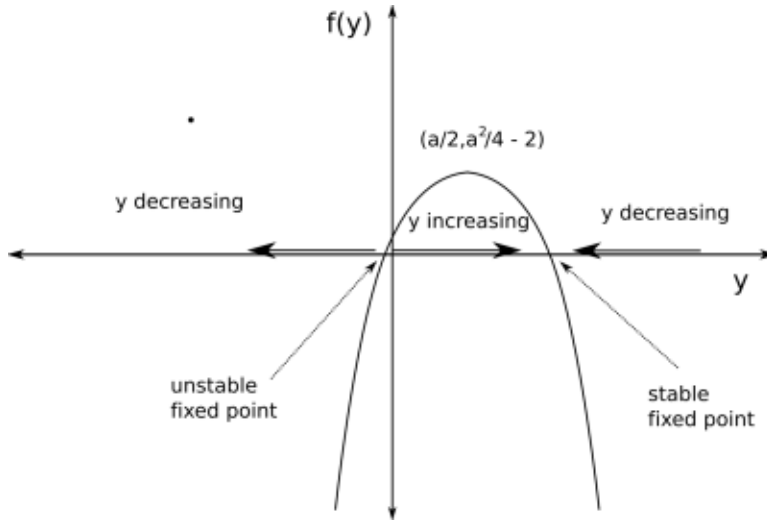


Figure 2: Phase diagram & stability (Note y should be x)

This characterizes the behavior of the solutions. If they start in between critical points, they will proceed left or right according to the arrows, and asymptotically approach either the next critical point, or $\pm\infty$ as $t \rightarrow \infty$.

We classify critical points as

- **stable** if solutions approach the critical point from either side
- **unstable** if solutions depart from the critical point on either side
- **semistable** if solutions approach from one direction and depart on another

Example. For what values of a does $x' = ax - x^2 - 2$ have a stable, positive solution?

Solution. Complete the square on the right hand side to get

$$ax - x^2 - 2 = -\left(x - \frac{a}{2}\right)^2 + \frac{a^2}{4} - 2$$

If a is large enough, there are two critical points on the x -axis, corresponding to the points where the graph of $f(x)$ crosses it (since by definition, this is where $f(x) = 0$). From the plot (2), it is clear that unless $a^2/4 > 2$, the parabola will lie completely below the x axis and there will be no fixed points at all.

The lower root is an unstable fixed point, and the upper one is stable. We therefore need a condition on a for which the larger root exists and is positive. The solutions to $-(x - \frac{a}{2})^2 + \frac{a^2}{4} - 2 = 0$ are

$$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 2}$$

and we are interested in the larger root $x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - 2}$. Thus the requirements on a that this exist (i.e. not be complex) and be positive are

$$\begin{aligned} \frac{a}{2} + \sqrt{\frac{a^2}{4} - 2} &> 0 \\ \frac{a^2}{4} &> 2 \end{aligned}$$

□

2.4 Numerical Methods

The basic numerical method for differential equations is **Euler's method**. Although much more sophisticated methods are used in practice, this is the simplest and the one we require students to know for this class. The idea is as follows. Suppose we have the initial value problem

$$x' = f(x, t), \quad x(t_0) = x_0$$

and we want to approximate the value of the solution $x(t)$ at a point t_n .

1. Divide the interval $[t_0, t_n]$ into n even subintervals of width h called the *step size*, to get

$$t_0 < t_1 < \cdots < t_n, \quad t_i - t_{i-1} = h$$

2. Start with the initial point $x_0 = x(t_0)$. We approximate the equation by

$$(x_1 - x_0)/h \approx f(x_0, t_0)$$

so we iteratively solve for

$$x_i = x_{i-1} + f(x_{i-1}, t_{i-1})h$$

3. Stop when you get to x_n

Euler's method will tend to *underestimate* solutions when they are upward sloping, and *overestimate* when the solutions are downward sloping.

Example. Estimate $x(2)$, where $x(t)$ is the solution to

$$x' = xt, \quad x(0) = 1$$

using a step size of $h = 0.5$.

Solution. It is best to make a table as follows

n	t_n	x_n	$f(x_n, t_n)$	$hf(x_n, t_n)$
0	0	1	0	0
1	1/2	$1 = 1 + 0$	1/2	1/4
2	1	$5/4 = 1 + 1/4$	5/4	5/8
3	3/2	$15/8 = 5/4 + 5/8$	45/16	45/32
4	2	$105/32 = 15/8 + 45/32$		

so we compute $x(2) \approx 105/32 = 3.2812$. This is likely to be an *underestimate* since the solution is increasing exponentially. Indeed, the exact solution is

$$x(t) = e^{t^2/2} \implies x(2) = e^2 = 7.4$$

□

3 Higher Order Linear Equations

This section will review the methods for solving higher order constant coefficient linear equations. See section 1 for definitions and terminology.

To solve an n th order constant coefficient linear initial value problem

$$p(D)x = x^{(n)} + a_1x^{(n-1)} + \cdots + a_0x = f(t), \quad x(0) = v_1, x'(0) = v_2, \dots, x^{(n-1)}(0) = v_n$$

the steps are as follows.

1. Solve the associated homogeneous problem $p(D)x = 0$. There will be n linearly independent solutions $x_1(t), \dots, x_n(t)$, and the homogeneous solution is given by $x_h(t) = c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t)$. The solutions are determined by the characteristic roots of the operator as described below in section 3.1.
2. Find a particular solution $x_p(t)$ to $p(D)x = f(t)$, by using one of the methods in section 3.2, such as undetermined coefficients or Fourier series.
3. The general solution is given by

$$x(t) = x_h(t) + x_p(t) = c_1x_1(t) + \dots + c_nx_n(t) + x_p(t)$$

where the two parameters c_1, \dots, c_n are yet unspecified.

4. Use the initial conditions $x(0) = v_1, \dots, x^{(n-1)}(0) = v_n$ to determine c_1, \dots, c_n and find the unique solution to the initial value problem.

Remark. If the equation is homogeneous to begin with, omit step 2.

Alternatively, you can solve the initial value problem using the Laplace transform (section 3.3, or by converting the equation to a first order system (section 4).

3.1 Homogeneous Solutions

Given a differential operator $p(D)$, the solutions to the homogeneous equation

$$p(D)x = (a_0D^n + a_1D^{n-1} \dots + a_n)x = 0$$

correspond to the roots of the associated **characteristic polynomial**

$$p(s) = a_0s^n + \dots + a_n = 0$$

It will have n roots r_1, \dots, r_n , some of which may be complex. Assuming the roots are distinct, we have corresponding independent solutions

$$x_i(t) = e^{r_i t}, \quad i = 1, \dots, n$$

We typically write complex exponentials in terms of sines and cosines. The case of repeated roots is discussed below.

In the case of the second order equation

$$x'' + bx' + cx = (D^2 + bD + c)x = 0$$

we have roots (from the quadratic formula)

$$s = \frac{1}{2} \left(-b \pm \sqrt{b^2 - 4c} \right)$$

There are three cases to consider.

1. r_1 **and** r_2 **are both real**, and $r_1 \neq r_2$. In this case the solutions are $e^{r_1 t}$ and $e^{r_2 t}$ and

$$x_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2. $r_1 = r_2$ **is a repeated root**. Write $r = r_1 = r_2$. Then the solutions are e^{rt} and te^{rt} . (In general, for an n th order ODE, if a root is a repeated root with multiplicity k , the solutions will be given by $t^j e^{rt}, j = 0, \dots, k$). So

$$x_h(t) = e^{rt}(c_1 + c_2 t)$$

3. r_1, r_2 are **complex**, and therefore given by $a \pm ib$. Then the corresponding real valued solutions are $e^{at} \cos bt$ and $e^{at} \sin bt$, and

$$x_h(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

If the real part $a = 0$ is equal to zero, the homogeneous solution is **periodic**. If the real part is nonzero $a \neq 0$, the solutions are not periodic, but we still refer to b as the **pseudofrequency**, with corresponding **pseudoperiod** $2\pi/b$.

Example. Find the general solution to the homogeneous ODE $x''' - 6x'' + 11x' = 0$.

Solution. We first write the equation as

$$(D^3 - 6D^2 + 11D)x = 0$$

so $p(s) = s^3 - 6s^2 + 11s = s(s^2 - 6s + 11)$. The roots are $s = 0$ and $s = r_{\pm}$, where

$$r_{\pm} = \frac{1}{2}(6 \pm \sqrt{36 - 44}) = \frac{1}{2}(6 \pm 2\sqrt{2}i) = 3 \pm \sqrt{2}i$$

So $x_h(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t)$ with

$$\begin{aligned} x_1(t) &= 1 \\ x_2(t) &= e^{3t} \cos \sqrt{2}t \\ x_3(t) &= e^{3t} \sin \sqrt{2}t \end{aligned}$$

so $x_2(t)$ and $x_3(t)$ have pseudofrequency $\sqrt{2}$. □

Example. Find the general solution to the homogeneous ODE $x'' + 6x' + 9x = 0$.

Solution. We first write the equation as

$$(D^2 + 6D + 9)y = 0$$

and the characteristic root equation is $p(s) = s^2 + 6s + 9 = (s + 3)^2 = 0$ and has the double root $s = -3$. Hence the homogeneous solution will be

$$x_h(t) = c_1e^{-3t} + c_2te^{-3t}$$

□

3.2 Particular Solutions

For certain types of input functions $f(t)$, we have direct methods for finding a particular solution.

3.2.1 Exponential Response

In the case that the input is exponential

$$f(t) = e^{\alpha t}, \quad \alpha \text{ a complex number}$$

then the particular solution will also have the form of an exponential. Its exact form depends on the differential operator. Suppose then, that we seek a particular solution to

$$p(D)x = e^{\alpha t}$$

The **exponential response formula** gives the result as follows

- **ERF**: $p(\alpha) \neq 0$. In this case, a particular solution will be given by

$$x_p(t) = \frac{e^{\alpha t}}{p(\alpha)}$$

- **ERF'**: $p(\alpha) = 0$ but $p'(\alpha) \neq 0$, where $p'(s) = \frac{d}{ds}p(s)$. In this case, a particular solution is given by

$$x_p(t) = \frac{te^{\alpha t}}{p'(\alpha)}$$

- **ERF^k**: $p(\alpha) = p'(\alpha) = \dots = p^{(k-1)}(\alpha) = 0$, but $p^{(k)}(\alpha) \neq 0$. Then a particular solution is

$$x_p(t) = \frac{t^k e^{\alpha t}}{p^{(k)}(\alpha)}$$

Example. Given the ODE $x'' + 2x' - 3x = f(t)$, find a particular solution when $f(t) = e^{2t}$ and when $f(t) = e^{-3t}$.

Solution. First, we write $x'' + 2x' - 3x = (D^2 + 2D - 3)x$. The roots of $p(s) = (s + 3)(s - 1)$ are 1 and -3 . In the first case, $f(t) = e^{2t}$ we have $p(2) \neq 0$, so ERF applies.

$$x_p(t) = \frac{e^{2t}}{(2^2 + 2 \cdot 2 - 3)} = \frac{e^{2t}}{5}$$

In the second case, $\alpha = -3$ is a single root, and so $p(-3) = 0$, but $p'(-3) = 2(-3) + 2 \neq 0$, and we get

$$x_p(t) = \frac{te^{-3t}}{2(-3) + 2} = -\frac{te^{-3t}}{4}$$

□

When $f(t) = \sin bt$ or $\cos bt$, we consider $\cos bt = \operatorname{Re}(e^{ibt})$ and $\sin bt = \operatorname{Im}(e^{ibt})$ and proceed as above. For a general product of exponentials with sines or cosines, we use $e^{at} \cos bt = \operatorname{Re}(e^{(a+ib)t})$, and can apply the exponential response formula where $\alpha = a + ib$. Remember to take the real or imaginary part of the resulting $x_p(t)$ to get the correct solution!

Example. Find a particular solution to $x'' - 4x' + 5x = e^{2t} \sin t$.

Solution. The characteristic polynomial associated to $p(D)$ is

$$p(s) = s^2 - 4s + 5$$

Using the quadratic formula, we have the roots

$$s = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = 2 \pm i$$

Writing $f(t) = \operatorname{Im}(e^{(2+i)t})$, we see that the coefficient in the exponent is a single root so we can use the ERF' formula. Taking a derivative of p we get $p'(s) = 2s - 4$. Thus

$$\begin{aligned} x_p(t) &= \operatorname{Im} \left(\frac{te^{(2+i)t}}{2(2+i) - 4} \right) = \operatorname{Im} \left(\frac{te^{(2+i)t}}{2i} \right) \\ &= \operatorname{Im} \left(\frac{(-2i)te^{(2+i)t}}{4} \right) = \operatorname{Im} \left(\frac{te^{2t}}{4} (-2i \cos t + 2 \sin t) \right) = -\frac{te^{2t} \cos t}{2} \end{aligned}$$

□

3.2.2 Undetermined Coefficients

When the input has the form of a polynomial times a (possibly complex) exponential,

$$f(t) = q(t)e^{\alpha t}, \quad q(t) \text{ a polynomial}$$

the particular solution will have the same form; however, we may have to shift the degree of the resulting polynomial depending on whether or not $p(\alpha), p'(\alpha)$, etc., vanish. In any case, we “guess” $x_p(t)$ to have the corresponding form, with undetermined coefficients, which we solve for by plugging $x_p(t)$ into the equation.

The method goes by the name of **undetermined coefficients**. Suppose $q(t)$ is a degree k polynomial (it’s highest power term is t^k). If

- $p(\alpha) \neq 0$: then we guess $x_p(t)$ to be

$$x_p(t) = (\text{general degree } k \text{ poly})e^{\alpha t} = (a_0t^k + a_1t^{k-1} + \dots + a_{k-1}t + a_k)e^{\alpha t}$$

where a_0, \dots, a_k are constants to be determined by plugging $x_p(t)$ into the equation.

- $p(\alpha) = 0$, but $p'(\alpha) \neq 0$: then we guess

$$x_p(t) = (\text{general degree } k \text{ poly})te^{\alpha t} = (a_0t^{k+1} + a_1t^k + \dots + a_{k-1}t^2 + a_kt)e^{\alpha t}$$

and solve for a_0, \dots, a_k by plugging in.

- $p(\alpha) = p'(\alpha) = \dots = p^{(l-1)}(\alpha) = 0$, but $p^{(l)}(\alpha) \neq 0$: then we guess

$$x_p(t) = (\text{general degree } k \text{ poly})t^le^{\alpha t} = (a_0t^{k+l} + a_1t^{k+l-1} + \dots + a_{k-1}t^{l+1} + a_kt^l)e^{\alpha t}$$

and solve for a_0, \dots, a_k by plugging in.

Remark. Note that the cases are the same as the exponential response formula. In fact, ERF is a special case of undetermined coefficients where $q(t)$ is a polynomial of degree 0; in this case ERF tells you what the resulting undetermined coefficient a_0 is, namely $a_0 = p(\alpha)^{-1}$, or $p'(\alpha)^{-1}$, etc.

Note also that undetermined coefficients also includes the case where $f(t) = q(t)$ is just a polynomial. In this case we take $\alpha = 0$ (since indeed, $q(t) = q(t)e^{0 \cdot t}$), so the particular solution to guess depends on to what degree 0 is a root of $p(s)$.

Example. Find a particular solution to $x'' + 3x' + Ax = t^2 + 1$ when $A = 2$ and when $A = 0$.

Solution. Start with $A = 2$. We have $p(s) = s^2 + 3s + 2$, and since the input has the form $(t^2 + 1)e^{0t}$, and $p(0) \neq 0$, we guess a solution of the form

$$x_p(t) = at^2 + bt + c$$

where a, b and c are arbitrary. When evaluating the left hand side on $x_p(t)$, consider organizing your computation as follows:

$$\left. \begin{array}{l} 2x_p \\ +3x'_p \\ +x''_p \\ = t^2 + 1 \end{array} \right\} = \left\{ \begin{array}{l} 2 \left(at^2 + bt + c \right) \\ +3 \left(2at + b \right) \\ +1 \left(2a \right) \\ = (2a)t^2 + (2b + 6a)t + (2c + 3b + 2a) \end{array} \right.$$

We conclude

$$2at^2 + (2b + 6a)t + (2c + 3b + 2a) = t^2 + 1, \quad \implies \quad a = \frac{1}{2}, \quad b = -\frac{3}{2}, \quad c = \frac{9}{4}$$

Thus, for $A = 2$, $x_p(t) = \frac{1}{4}(2t^2 - 6t + 9)$.

When $A = 0$, $p(s) = s^2 + 3s$, and we have $p(0) = 0$ but $p'(0) \neq 0$. So we guess a solution of the form

$$x_p(t) = at^3 + bt^2 + ct$$

and we compute

$$\left. \begin{array}{l} 3x'_p \\ + x''_p \\ = t^2 + 1 \end{array} \right\} = \left\{ \begin{array}{l} 3 \left(\begin{array}{l} 3at^2 \\ + \\ 6at \\ + \\ 2b \end{array} \right) \\ + 1 \left(\begin{array}{l} 2bt \\ + \\ 3c + 2b \end{array} \right) \\ = (9a)t^2 + (6b + 6a)t + (3c + 2b) \end{array} \right.$$

We get

$$a = \frac{1}{9}, \quad b = -\frac{1}{9}, \quad c = \frac{11}{27}$$

So for $A = 0$, $x_p(t) = \frac{1}{9}(t^3 - t^2 + \frac{11}{3}t)$. □

Example. Find a particular solution to

$$x'' + 7x' + 10x = 6te^{-2t}$$

Solution. We have $p(s) = s^2 + 7s + 10 = (s + 2)(s + 5)$ so $p(-2) = 0$ but $p'(-2) \neq 0$. Since $q(t) = t$ is a degree 1 polynomial, we guess $x_p(t)$ of the form

$$x_p(t) = (at + b)te^{-2t} = (at^2 + bt)e^{-2t}$$

We plug this into the equation to solve for a and b . It is convenient to organize the computation as follows (remember also to include the effect of differentiating the e^{-2t} when computing the derivatives!)

$$\left. \begin{array}{l} 10x_p \\ + 7x'_p \\ + 1x''_p \\ = 6te^{-2t} \end{array} \right\} = \left\{ \begin{array}{l} 10 \left(\begin{array}{l} at^2 \\ + \\ bt \\ + \\ 0 \end{array} \right) e^{-2t} \\ + 7 \left(\begin{array}{l} -2at^2 \\ + \\ (2a - 2b)t \\ + \\ b \end{array} \right) e^{-2t} \\ + 1 \left(\begin{array}{l} 4at^2 \\ + \\ (-4a - 2(2a - 2b))t \\ + \\ ((2a - 2b) - 2b) \end{array} \right) e^{-2t} \\ = \left(\begin{array}{l} 0t^2 \\ + \\ 6at \\ + \\ (2a + 3b) \end{array} \right) e^{-2t} \end{array} \right.$$

so we must have

$$6a = 6, \quad (2a + 3b) = 0 \implies a = 1, \quad b = -2/3$$

We conclude that

$$x_p(t) = (t^2 - 2/3t)e^{-2t}$$

is a particular solution. □

3.2.3 Fourier Series for Periodic Inputs

When the input function to a linear equation is **periodic**, that is, $f(t+P) = f(t)$ for some **period** P , we can use Fourier series to construct a particular solution. For general techniques of Fourier series, see appendix B; here we will focus on the application to particular solutions.

The idea is to express $f(t)$ by its Fourier series, and then construct the particular solution one term at a time. Suppose we want to solve

$$p(D)x = f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

We let $x_{a_n}(t)$ and $x_{b_n}(t)$ be the particular solutions to the following ODE

$$\begin{aligned} p(D)x_{a_0} &= \frac{a_0}{2} \\ p(D)x_{a_n} &= a_n \cos\left(\frac{n\pi t}{L}\right) \\ p(D)x_{b_n} &= b_n \sin\left(\frac{n\pi t}{L}\right) \end{aligned}$$

Then, by the additive properties of linear differential equations (see section 1), a particular solution to the original problem will be given by

$$x_p(t) = x_{a_0}(t) + \sum_{n=1}^{\infty} x_{a_n}(t) + \sum_{n=1}^{\infty} x_{b_n}(t)$$

Remark. Depending on the operator $p(D)$, it may be the case that each particular solution $x_{a_n}(t)$ and $x_{b_n}(t)$ has the form of sines, cosines and constant terms. If this happens, then the sum $x_p(t)$ is itself a Fourier series and so the response will also be *periodic*. If however, *any one* of the particular solutions $x_{a_n}(t)$ or $x_{b_n}(t)$ has a different form (such as $t \cos(n\pi t/L)$ from the ERF' formula when $i(n\pi/L)$ is a root of $p(s)$), then the resulting $x_p(t)$, while still a particular solution, will *not* be periodic.

Example. Find a particular solution to $x'' + kx = f(t)$, where $f(t)$ is the odd periodic extension of t with period 4 (a sawtooth wave). For what k will the solution be periodic?

Solution. The Fourier series for $f(t)$ is computed in an example in section B. We obtain

$$f(t) = \sum_n b_n \sin\left(\frac{n\pi t}{2}\right), \quad b_n = \frac{4(-1)^{n+1}}{n\pi}$$

We proceed term by term, seeking a particular solution to the equation

$$x_n'' + kx_n = b_n \sin\left(\frac{n\pi t}{2}\right)$$

For this we can use the exponential response formula (3.2.1), thinking of $x_n = \text{Im}(\tilde{x}_n)$, where

$$\tilde{x}_n'' + k\tilde{x}_n = b_n e^{i(n\pi/2)t}$$

Provided $p(i(n\pi/2)) = k - (n\pi/2)^2 \neq 0$, we can use ERF, to obtain

$$\tilde{x}_n(t) = \frac{b_n}{k - (n\pi/2)^2} e^{i(n\pi/2)t} \implies x_n(t) = \frac{b_n}{k - (n\pi/2)^2} \sin\left(\frac{n\pi t}{2}\right)$$

So the condition which guarantees a periodic *particular* solution is

$$k \neq \left(\frac{n\pi}{2}\right)^2, \quad \text{for any } n$$

in which case

$$x_p(t) = \sum_n \tilde{b}_n \sin\left(\frac{n\pi t}{2}\right), \quad \tilde{b}_n = \frac{b_n}{k - (n\pi/2)^2} = \frac{4(-1)^{n+1}}{n\pi(k - (n\pi/2)^2)}$$

Since the characteristic polynomial $p(s) = s^2 + k = (s + ik)(s - ik)$ has purely imaginary roots, the *general* solution will also be periodic in this case.

However, if for example, $k = \pi^2/4$, so that it violates our condition when $n = 1$, the solution will *not* be periodic, since we obtain

$$x_p(t) = -\frac{t}{\pi} \cos\left(\frac{\pi t}{2}\right) + \sum_{n=2}^{\infty} \tilde{b}_n \sin\left(\frac{n\pi t}{2}\right)$$

where the **resonant**³ term $x_1(t)$ was calculated using ERF':

$$\begin{aligned} x_1(t) &= \text{Im}\left(\frac{t}{p'(i\pi/2)} e^{i(\pi/2)t}\right) = \text{Im}\left(\frac{t}{2i(\pi/2)} e^{i(\pi/2)t}\right) \\ &= \text{Im}\left(\frac{-it}{\pi} (\cos(\pi t/2) + i \sin(\pi t/2))\right) = -\frac{t}{\pi} \cos\left(\frac{\pi t}{2}\right) \end{aligned}$$

□

³**Resonance** is the term we use for the response of a system of the form $x'' + \omega_0^2 x = f(t)$ to an oscillator input $f(t)$ which has the same frequency ω_0 as the homogeneous solutions. The particular solution (called the resonant) solution, grows without bound because of the t in the amplitude.

3.3 Laplace Transform Methods

The Laplace transform is a useful tool for solving initial value problems for a few reasons.

1. The transform itself automatically takes initial conditions into account, resulting in a particular solution satisfying the initial data without the need to go through a general solution first.
2. It is well suited to handle input functions which are **piecewise**, that is,

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1 \\ f_2(t) & t_1 < t < t_2 \\ \vdots & \vdots \end{cases}$$

3. In the case of purely zero initial conditions, solving by weight functions and convolution as in section 3.3.1 allows us to write down a direct general formula for the solution to an IVP with arbitrary $f(t)$.

For general results and techniques regarding the Laplace transform, see appendix C. Here we will focus on its use in solving initial value problems.

Example. Use the Laplace transform to solve the initial value problem

$$x'' + 7x' + 10x = 6te^{-2t}, \quad x(0) = 0, x'(0) = 2$$

Solution. Taking the Laplace transform of both sides of the equation, we get

$$s^2 X(s) - sx(0) - x'(0) + 7(sX(s) - x(0)) + 10X(s) = (s^2 + 7s + 10)X(s) - 2 = \frac{6}{(s+2)^2}$$

Note that the coefficient in front of $X(s)$ is $p(s) = s^2 + 7s + 10 = (s+2)(s+5)$, as will always happen. We solve for $X(s)$ to get

$$X(s) = \frac{1}{(s+2)(s+5)} \left(\frac{6}{(s+2)^2} + 2 \right) = \frac{1}{(s+2)(s+5)} \left(\frac{6}{(s+2)^2} + \frac{2(s+2)^2}{(s+2)^2} \right) = \frac{6 + 2(s+2)^2}{(s+2)^3(s+5)}$$

We must use partial fraction decomposition to write⁴

$$\frac{6 + 2(s+2)^2}{(s+2)^3(s+5)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{s+5}$$

Heaviside coverup (see appendix C) applies to C and D to give

$$C = \frac{6 + 2(-2+2)^2}{(-2+5)} = \frac{6}{3} = 2, \quad D = \frac{6 + 2(-5+2)^2}{(-5+2)^3} = \frac{6+18}{-27} = -\frac{8}{9}$$

To obtain A and B we must put everything over a common denominator and compare sides (it's best not to expand everything out yet):

$$A(s+2)^2(s+5) + B(s+2)(s+5) + C(s+5) + D(s+2)^3 = 6 + 2(s+2)^2$$

Comparing terms of order s^3 , we get

$$A + D = A - 8/9 = 0 \implies A = \frac{8}{9}$$

⁴This is a more complicated partial fraction decomposition than you would have to do on an exam. We do it for this problem so that we can compare with the solution obtained by earlier methods.

Comparing terms of order s^2 (count very carefully!), we get

$$5A + 2(2A) + B + 3(2D) = B + 9A + 6D = B + 3(8/9) = 2 \implies B = -\frac{6}{9} = -\frac{2}{3}$$

So

$$X(s) = \frac{8}{9} \frac{1}{s+2} - \frac{2}{3} \frac{1}{(s+2)^2} + \frac{2}{(s+2)^3} - \frac{8}{9} \frac{1}{s+5}$$

and the inverse transform gives our solution

$$x(t) = \frac{8}{9}e^{-2t} - \frac{2}{3}te^{-2t} + t^2e^{-2t} - \frac{8}{9}e^{-5t}$$

Note that the particular solution $x_p(t) = (t^2 - 2/3t)e^{-2t}$ for this operator was computed by the method of undetermined coefficients in section 3.2.2. We can then easily check our Laplace transform answer with what we would get via the general solution. Since $p(s) = (s+2)(s+5)$, the general solution will be

$$x_g(t) = c_1e^{-2t} + c_2e^{-5t} + (t^2 - 2/3t)e^{-2t}$$

Matching the initial conditions $x(0) = 0$, $x'(0) = 2$ gives $c_1 = 8/9$, $c_2 = -8/9$, which agrees with the above. \square

Example. Find the solution to

$$x'' + x = \begin{cases} t & 0 < t < \pi/2 \\ 0 & t > \pi/2 \end{cases} \quad x(0) = x'(\pi/2) = 0$$

Solution. The input function is piecewise; we can write it in terms of step functions using the methods of appendix C.

$$f(t) = t(u(t) - u(t - \pi/2))$$

We take the Laplace transform to get

$$\begin{aligned} (s^2 + 1)X(s) &= \mathcal{L}\left(tu(t) - u(t - \pi/2)((t - \pi/2) + \pi/2)\right) = \frac{1}{s^2} - e^{-\pi/2s} \left(\frac{1}{s^2} + \frac{\pi/2}{s}\right) \\ X(s) &= \frac{1}{s^2(s^2 + 1)} - e^{-\pi/2s} \left(\frac{1}{s^2(s^2 + 1)} + \frac{\pi/2}{s(s^2 + 1)}\right) \end{aligned}$$

We must do two partial fraction decompositions. First,

$$\begin{aligned} \frac{1}{s^2(s^2 + 1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1} \\ 1 &= As(s^2 + 1) + B(s^2 + 1) + (Cs + D)s^2 \\ \left. \begin{array}{l} s^3 : \quad 0 &= A + C \\ s^2 : \quad 0 &= B + D \\ s : \quad 0 &= A \\ 1 : \quad 1 &= B \end{array} \right\} \implies A = C = 0, \quad B = 1, \quad D = -1 \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\pi/2}{s(s^2 + 1)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \\ \pi/2 &= A(s^2 + 1) + (Bs + C)s \\ \left. \begin{array}{l} s^2 : \quad 0 &= A + B \\ s : \quad 0 &= C \\ 1 : \quad \pi/2 &= A \end{array} \right\} \implies C = 0, \quad A = \pi/2, \quad B = -\pi/2 \end{aligned}$$

Thus, we get

$$\begin{aligned}
 X(s) &= \frac{1}{s^2} - \frac{1}{s^2 + 1} - e^{-\pi/2s} \left(\frac{1}{s^2} - \frac{1}{s^2 + 1} + \frac{\pi/2}{s} - (\pi/2) \frac{s}{s^2 + 1} \right) \\
 x(t) &= t - \sin(t) - u(t - \pi/2) \left((t - \pi/2) - \sin(t - \pi/2) + \pi/2 - \pi/2 \cos(t - \pi/2) \right) \\
 &= t - \sin(t) - u(t - \pi/2) \left(t + \cos(t) - \pi/2 \sin(t) \right) \\
 x(t) &= \begin{cases} t - \sin t & 0 < t < \pi/2 \\ (\pi/2 - 1) \sin(t) - \cos(t) & t > \pi/2 \end{cases}
 \end{aligned}$$

□

3.3.1 Weight Function, Transfer Function, & Convolution

To any given differential operator $p(D)$, we can associate the corresponding **weight function**, which is the solution to the initial value problem

$$p(D)w(t) = \delta(t), \quad w(0) = w'(0) = \dots = 0$$

where $\delta(t)$ is the **Dirac delta function**. It is important here that the initial conditions are identically 0, sometimes referred to as **rest initial conditions**. Taking the Laplace transform of both sides yields

$$p(s)W(s) = 1 \quad \implies \quad \mathcal{L}(w(t)) = W(s) = \frac{1}{p(s)}$$

We call $W(s)$ the **transfer function**. Note that knowing the weight function is equivalent to knowing the differential operator, since we can compute $p(D)$ from $p(s) = 1/W(s)$.

Remark. An equivalent characterization of the weight function is as the solution to the initial value problem

$$p(D)w(t) = 0, \quad w(0) = w'(0) = \dots = w^{(n-2)}(0) = 0, \quad w^{(n-1)}(0) = 1/a_0$$

where n is the order of the equation, and a_0 is the top order coefficient of $p(D) = a_0 D^n + \dots$. This follows by the Laplace transform, since the left hand side will have a term $-a_0 w^{(n-1)}(0)$ coming from the transform of the highest derivative.

The main feature of the weight function is that, once we know $w(t)$ for a given operator $p(D)$, we can solve the corresponding initial value problem (with rest initial conditions)

$$p(D)x(t) = f(t), \quad x(0) = x'(0) = \dots = 0$$

for any input $f(t)$, by taking the **convolution** (see appendix C.1)

$$x(t) = (w * f)(t) = \int_0^t w(\tau) f(t - \tau) d\tau$$

Thus, we have a direct formula for the solution with arbitrary input function without having to re-solve the equation every time.

Example. Write an integral formula in terms of $f(t)$ for the solution to

$$x'' + 6x' + 9x = f(t), \quad x(0) = x'(0) = 0$$

Solution. We first determine $w(t)$, which satisfies

$$w'' + 6w' + 9w = \delta(t), \quad w(0) = w'(0) = 0 \quad \text{or} \quad w'' + 6w' + 9w = 0, \quad w(0) = 0, w'(0) = 1$$

by our favorite methods. We will do this two ways. Using the Laplace transform, and $p(s) = s^2 + 6s + 9 = (s + 3)^2$, we have

$$w(t) = \mathcal{L}^{-1} \left(\frac{1}{(s + 3)^2} \right) = te^{-3t}$$

Alternatively, we can use the characteristic roots (-3 , repeated root), to obtain the general solution to $w'' + 6w' + 9w = 0$,

$$w_g(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

The initial conditions $w(0) = 0$ and $w'(0) = 1$ force $c_1 = 0$, and

$$w'(0) = c_2 = 1$$

In any case, then the solution to the original equation in x can be written as

$$x(t) = w(t) * f(t) = \int_0^t \tau e^{-3\tau} f(t - \tau) d\tau = \int_0^t f(\tau)(t - \tau) e^{-3(t-\tau)} d\tau = f(t) * w(t)$$

□

Example. Suppose a solution to

$$p(D)w(t) = \delta(t), \quad w(0) = w'(0) = \dots = 0$$

is given by

$$w(t) = e^{-2t}$$

where $p(D)$ is a constant coefficient linear differential operator. Find the solution to the initial value problem

$$p(D)x(t) = e^{-t}, \quad x(0) = x'(0) = \dots = 0$$

Solution. Even though we don't know the operator, we can write down the solution as

$$\begin{aligned} x(t) &= w(t) * e^{-t} = e^{-2t} * e^{-t} \\ &= \int_0^t e^{-2s} e^{-(t-s)} ds = e^{-t} \int_0^t e^{-s} ds \\ &= -e^{-t} e^{-s} \Big|_0^t = -e^{-2t} + e^{-t} \end{aligned}$$

since we have rest initial conditions.

Alternatively, using the Laplace transform, we can recover $p(D)$ by

$$W(s) = \frac{1}{p(s)} = \mathcal{L}(e^{-2t}) = \frac{1}{s + 2}$$

so

$$p(D) = D + 2$$

and we can solve by any other method. □

4 Systems of Equations

See section 1 for definitions and terminology. We deal mostly with linear systems

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{r}(t)$$

The steps for solving such an equation are analogous to the ones for higher order linear equations discussed in section 3.1, namely

1. Find the **homogeneous solution** $\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$, as described in section 4.2, where $\mathbf{x}_i(t)$ solves $\mathbf{x}'_i = A\mathbf{x}_i$.
2. (If applicable), find a **particular solution** $\mathbf{x}_p(t)$ to $\mathbf{x}'_p = A\mathbf{x}_p + \mathbf{r}$, as described in section 4.4. Then the **general solution** is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t)$$

3. (If applicable), Given an initial condition $\mathbf{x}(0) = \mathbf{x}_0$, solve for the constants c_i to get the solution to the **initial value problem**.

For nonlinear systems, the best we can do is get a qualitative picture of the solutions using the graphical methods of section 4.6.

4.1 Relationship to Higher Order Equations

You can always convert back and forth between n th order constant coefficient linear equations and first order n -dimensional systems of ODEs. Starting with the n th order equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = f(t)$$

define new variables (which will be the vector components)

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

Taking derivatives, we have by definition for the first $n - 1$ variables,

$$\begin{aligned} x'_1 &= y' = x_2 \\ x'_2 &= x_3 \\ &\vdots \\ x'_{n-1} &= x_n \end{aligned}$$

The derivative of x_n is $y^{(n)}$, which we can solve for in the original equation, replacing derivatives of y by their corresponding x_i variables

$$x'_n = -a_1x_n - \cdots - a_nx_1 + f(t)$$

So

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t) = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}$$

A is called the **companion matrix** to the original equation.

Example. Convert $y'' + 3y' + 2y = e^t$ to a two dimensional system

Solution. We have

$$\begin{aligned}x_1 &= y \\x_2 &= y'\end{aligned}$$

So

$$\begin{aligned}x_1' &= y' = x_2 \\x_2' &= y'' = -3y' - 2y + e^t = -3x_2 - 2x_1 + e^t\end{aligned}$$

Which we can write as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

□

We illustrate the conversion from a 2 dimensional system to a second order equation by example.

Example. Convert the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

into a single, second order equation

Solution. Choose one of the two variables which will be the second order variable, say x_1 . Start by taking the derivative of the corresponding equation

$$x_1'' = 3x_1' + x_2' + 2t = 3x_1' + (-x_1 + x_2 + e^t) + 2t$$

where we substituted the other equation in for x_2' . We still need to get rid of the x_2 term, which we can do by using the first equation (we've only used the *derivative* of the first equation, not the equation itself, so this will not lead to any redundancy). Solving the first equation for x_2 and substituting, we get

$$x_1'' = 3x_1' + (-x_1 + (x_1' - 3x_1 - t^2) + e^t) + 2t = 4x_1' - 4x_1 - t^2 + 2t + e^t$$

Or, in more standard form

$$x_1'' - 4x_1' + 4x_1 = -t^2 + 2t + e^t$$

□

4.2 Homogeneous Solutions

Given a $n \times n$ system

$$\mathbf{x}' = A\mathbf{x}$$

we seek n linearly independent solutions $\mathbf{x}_1, \dots, \mathbf{x}_n$. Supposing these solutions to have the form of an exponential in t times a constant vector (in analogy with the higher order equations of section 3.1), we guess

$$\mathbf{x}(t) = e^{\alpha t} \mathbf{v}$$

and plug in.

$$(e^{\alpha t} \mathbf{v})' = \alpha e^{\alpha t} \mathbf{v} = A e^{\alpha t} \mathbf{v} \iff \alpha \mathbf{v} = A \mathbf{v}$$

We see that this will be a solution provided α is an **eigenvalue** with corresponding **eigenvector** \mathbf{v} . For a discussion of eigenvalues and eigenvectors, see appendix D.1.

To sum, the general solution will have the form

$$\mathbf{x}_h(t) = c_1 \mathbf{x}_1(t) + \dots + c_n \mathbf{x}_n(t)$$

where $\mathbf{x}_i(t)$ is a solution corresponding to eigenvalue λ_i . As for the individual solutions $\mathbf{x}_i(t)$, there are four cases to consider, and examples are provided below:

1. $\lambda_1, \dots, \lambda_n$ **are real and distinct**. In this case, there will be n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, each corresponding to an eigenvalue, and each individual solution has the form

$$\mathbf{x}_i(t) = \mathbf{v}_i e^{\lambda_i t}$$

2. **Some of $\lambda_1, \dots, \lambda_n$ are complex**. In this case, since the coefficients of A are real, the complex eigenvalues will come in conjugate pairs $\lambda_j = a + ib$, and $\lambda_k = a - ib$. To get two linearly independent solutions for λ_j and λ_k , it is sufficient to choose one of them, say $\lambda_j = a + ib$, and compute its eigenvector \mathbf{v}_j which will have complex entries. Then we can take

$$\mathbf{x}_j = \operatorname{Re} \left(\mathbf{v}_j e^{(a+ib)t} \right), \quad \mathbf{x}_k = \operatorname{Im} \left(\mathbf{v}_j e^{(a+ib)t} \right)$$

3. λ_i **is repeated with multiplicity k , but it has k independent eigenvectors**. This is called the **complete** case for repeated eigenvalues. In this case, the solution is identical to case 1, simply treating λ_i as k distinct eigenvalues. Thus if $\mathbf{v}_1, \dots, \mathbf{v}_k$ are its eigenvectors, we have k independent solutions

$$\mathbf{x}_{i_1} = \mathbf{v}_1 e^{\lambda_i t}, \quad \dots, \quad \mathbf{x}_{i_k} = \mathbf{v}_k e^{\lambda_i t}$$

4. ***(Will not be on exam!)* λ_i is repeated with multiplicity k , but it has fewer than k independent eigenvectors**. This is called the **defective** case for repeated eigenvalues. In this case we can find *chains* of *generalized* eigenvectors (see below). Say we have two independent eigenvectors \mathbf{v}_{1_1} and \mathbf{v}_{2_1} . Then we can find chains $\{\mathbf{v}_{1_1}, \dots, \mathbf{v}_{1_l}\}$ and $\{\mathbf{v}_{2_1}, \dots, \mathbf{v}_{2_m}\}$ where $l+m = k$, so there are a total of k generalized eigenvectors. The equations defining the chains of eigenvectors are $(A - \lambda I)\mathbf{v}_{1_l} = \mathbf{v}_{1_{l-1}}, \dots, (A - \lambda I)\mathbf{v}_{1_2} = \mathbf{v}_{1_1}, (A - \lambda I)\mathbf{v}_{1_1} = 0$; so we can start with the beginning of the chain, which is a true eigenvector, and inductively solve for the successive elements of the chain. Once we have done this, we have independent solutions

$$\begin{aligned} \mathbf{x}_{1_1} &= \mathbf{v}_{1_1} e^{\lambda_i t} \\ \mathbf{x}_{1_2} &= (\mathbf{v}_{1_1} t + \mathbf{v}_{1_2}) e^{\lambda_i t} \\ \mathbf{x}_{1_3} &= \left(\mathbf{v}_{1_1} \frac{t^2}{2} + \mathbf{v}_{1_2} t + \mathbf{v}_{1_3} \right) e^{\lambda_i t} \\ &\vdots \\ \mathbf{x}_{2_1} &= \mathbf{v}_{2_1} e^{\lambda_i t} \\ \mathbf{x}_{2_2} &= (\mathbf{v}_{2_1} t + \mathbf{v}_{2_2}) e^{\lambda_i t} \\ \mathbf{x}_{2_3} &= \left(\mathbf{v}_{2_1} \frac{t^2}{2} + \mathbf{v}_{2_2} t + \mathbf{v}_{2_3} \right) e^{\lambda_i t} \\ &\vdots \end{aligned}$$

Example. Find the general solution to $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}$$

Solution. We must compute the eigenvalues and eigenvectors of A as described in appendix D.1. To find the eigenvalues, we set

$$0 = \det(A - \lambda I) = \det \begin{vmatrix} 2 - \lambda & 4 \\ 0 & -1 - \lambda \end{vmatrix} = (2 - \lambda)(-1 - \lambda) = \lambda^2 - \lambda - 2$$

Alternatively, we can use the direct formula for 2×2 matrices given by (see appendix D.1)

$$\det(A - \lambda I) = \lambda^2 - \text{tr } A\lambda + \det A$$

which agrees with our answer. The eigenvalues are -1 and 2 , and so are distinct. To find the eigenvector corresponding to $\lambda = -1$, we solve into the equation $(A - \lambda I)\mathbf{v} = 0$ to get

$$(A - (-1)I)\mathbf{v} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 3v_1 + 4v_2 = 0$$

so we can take

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Similarly, for the eigenvalue $\lambda = 2$ we have

$$\begin{bmatrix} 0 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u_2 = 0$$

So we can take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 4 \\ -3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

□

Example. Find the general solution to $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$$

Solution. The characteristic polynomial is

$$\lambda^2 - 4\lambda + 8 = 0 \implies \lambda = 2 \pm 2i$$

We have complex eigenvalues. We proceed to find two independent real valued solutions as described above: we choose one of these eigenvalues, find its eigenvector, and take real and imaginary parts at the end to get independent solutions. So take $\lambda = 2 + 2i$. Substituting into $(A - \lambda I)\mathbf{v} = 0$ we have the equation

$$\begin{bmatrix} -1 - 2i & -5 \\ 1 & 1 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives us two equations which are the same (checking this is a good exercise in complex arithmetic), one is $v_1 + (1 - 2i)v_2 = 0$, and we can take the solution

$$\mathbf{v} = \begin{bmatrix} 1 - 2i \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Then we have

$$\begin{aligned} \mathbf{x}_1 &= \text{Re} \left(e^{2t} (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} \cos 2t + 2 \sin 2t \\ -\cos 2t \end{bmatrix} \\ \mathbf{x}_2 &= \text{Im} \left(e^{2t} (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} \sin 2t - 2 \cos 2t \\ -\sin 2t \end{bmatrix} \\ \mathbf{x} &= c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \end{aligned}$$

□

Example. Find the general solution to $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -4 & 6 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & -4 \end{bmatrix}$$

Solution. First we need to solve $\det(A - \lambda I) = 0$, which we calculate directly from the definition, using cofactor expansion (see appendix D) to compute the determinant

$$\det \begin{vmatrix} -4-\lambda & 6 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 6 & -4-\lambda \end{vmatrix} = (-4-\lambda)\det \begin{vmatrix} 2-\lambda & 0 \\ 6 & -4-\lambda \end{vmatrix} = (2-\lambda)(-4-\lambda)^2 = 0$$

For $\lambda = 2$, we compute

$$\begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

to get an eigenvector \mathbf{v}_1 . For $\lambda = -4$, we have the eigenvector equation

$$\begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which amounts to the single equation $v_2 = 0$. Since we have two degrees of freedom, we can find two independent eigenvectors, and the matrix is complete. We take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) \\ \mathbf{x}_1 &= e^{-2t}\mathbf{v}_1 = e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \mathbf{x}_2 &= e^{-4t}\mathbf{v}_2 = e^{-4t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_3 &= e^{-4t}\mathbf{v}_3 = e^{-4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

□

4.3 Fundamental Matrices & Initial Value Problems

Given an $n \times n$ matrix A , a **fundamental matrix** for A is an $n \times n$ matrix $F(t)$ satisfying

$$F'(t) = AF(t), \quad \det F(t) \neq 0 \text{ for all } t$$

Writing the columns of $F(t)$ as vectors $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$,

$$F(t) = \begin{bmatrix} | & \cdots & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & \cdots & | \end{bmatrix}$$

the differential equation for $F(t)$ amounts to the n equations

$$\mathbf{x}'_i(t) = A\mathbf{x}_i, \quad i = 1, \dots, n$$

and the condition $\det F(t) \neq 0$ is equivalent to $\mathbf{x}_1, \dots, \mathbf{x}_n$ linearly independent. Thus the following two things are equivalent

- $F'(t) = AF(t)$, $\det F(t) \neq 0$ for all t
- $F(t)$ has columns consisting of linearly independent solutions $\mathbf{x}_1, \dots, \mathbf{x}_n$ to the equation $\mathbf{x}' = A\mathbf{x}$.

Remark. $F(t)$ is not unique, as indeed we could consider rearranging its columns or multiplying them by various constants to get a new fundamental matrix.

In light of the above, we can express the general solution to $\mathbf{x}' = A\mathbf{x}$ as follows

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = F(t)\mathbf{c}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Suppose we want to find a solution satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Since $F(0)$ is invertible (because $\det F(0) \neq 0$), we can write

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{x}(0) = F(0)\mathbf{c} \\ \mathbf{c} &= F(0)^{-1}\mathbf{x}_0 \end{aligned}$$

Then

$$\mathbf{x}(t) = F(t)\mathbf{c} = F(t)F(0)^{-1}\mathbf{x}_0$$

is the solution to the initial value problem. Note that while fundamental matrices are not unique, $F(t)F(0)^{-1}$ is always the same matrix, even when computed by different fundamental matrices. In fact

$$F(t)F(0)^{-1} = e^{tA}, \quad \text{for any fundamental matrix } F(t)$$

where e^{tA} is a **matrix exponential**, discussed in appendix D.2. So we conclude

- The solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0$$

- We can compute e^{tA} for any matrix A from any fundamental matrix $F(t)$ for A by setting

$$e^{tA} = F(t)F(0)^{-1}$$

Example. Solve the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

Solution. Eigenvalues and eigenvectors for A are calculated in an example in appendix D.1. We have

$$\lambda_1 = -5, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \lambda_2 = 5, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus we have linearly independent homogeneous solutions

$$\mathbf{x}_1 = \begin{bmatrix} e^{-5t} \\ -2e^{-5t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2e^{5t} \\ e^{5t} \end{bmatrix}$$

and can form the fundamental matrix

$$F(t) = \begin{bmatrix} e^{-5t} & 2e^{5t} \\ -2e^{-5t} & e^{5t} \end{bmatrix}$$

and compute

$$\begin{aligned} F(0)^{-1} &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\ e^{tA} = F(t)F(0)^{-1} &= \frac{1}{5} \begin{bmatrix} e^{-5t} & 2e^{5t} \\ -2e^{-5t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} (-2e^{-5t} + 2e^{5t}) & (-2e^{-5t} + 2e^{5t}) \\ (-2e^{-5t} + 2e^{5t}) & (4e^{-5t} + e^{5t}) \end{bmatrix} \end{aligned}$$

The solution to the initial value problem is then

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 = \frac{1}{5} \begin{bmatrix} (-2e^{-5t} + 2e^{5t}) & (-2e^{-5t} + 2e^{5t}) \\ (-2e^{-5t} + 2e^{5t}) & (4e^{-5t} + e^{5t}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} (-2e^{-5t} + 2e^{5t}) \\ (-2e^{-5t} + 2e^{5t}) \end{bmatrix}$$

□

4.4 Inhomogeneous Equations & Variation of Parameters

We can also use a fundamental matrix to find particular solutions to an **inhomogeneous system**

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{r}(t)$$

by a method called **variation of parameters**. This method works for *any* input function⁵ $\mathbf{r}(t)$.

The trick is as follows. First compute a fundamental matrix $F(t)$ for A . Then guess a particular solution of the form

$$\mathbf{x}_p(t) = F(t)\mathbf{v}(t)$$

where $\mathbf{v}(t)$ is a yet-to-be-determined vector function of t . Plug this into the equation and use the product rule⁶ to get

$$(F(t)\mathbf{v}(t))' = F'(t)\mathbf{v}(t) + F(t)\mathbf{v}'(t) = A(F(t)\mathbf{v}(t)) + \mathbf{r}(t)$$

Since $F(t)$ is a fundamental matrix, we have $F'(t) = AF(t)$, so the first terms on each side of the equation cancel, and we get

$$F(t)\mathbf{v}'(t) = \mathbf{r}(t) \implies \mathbf{v}'(t) = F(t)^{-1}\mathbf{r}(t)$$

using the fact that $F(t)$ is invertible (since the other condition on a fundamental matrix is $\det F(t) \neq 0$ for all t). We integrate (the lower limit of integration is unimportant; different choices will lead to different particular solutions)

$$\mathbf{v}(t) = \int_0^t F(\tau)^{-1}\mathbf{r}(\tau) d\tau \implies \mathbf{x}_p(t) = F(t)\mathbf{v}(t) = F(t) \int_0^t F(\tau)^{-1}\mathbf{r}(\tau) d\tau$$

⁵For this reason, it is useful in the case of a single higher order inhomogeneous equation whose input function isn't amenable to the techniques of section 3.2. We can transform such an equation into an inhomogeneous first order system, find a particular solution, and transform back.

⁶It is straightforward (and a good exercise!) to check that a product rule applies to matrix multiplication

We emphasize that this formula is valid for any fundamental matrix $F(t)$. In particular, if we use the matrix exponential $F(t) = e^{tA}$, we have

$$\mathbf{x}_p(t) = e^{tA} \int_0^t e^{-\tau A} \mathbf{r}(\tau) d\tau$$

since $(e^{\tau A})^{-1} = e^{-\tau A}$ as discussed in appendix D.2. Another fundamental matrix may give an easier integral however.

Example. Find a particular solution to

$$\mathbf{x}'(t) = A\mathbf{x}(t) = \mathbf{r}(t), \quad A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \mathbf{r}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

Solution. First we find a fundamental matrix for A . Its eigenvalues and eigenvectors can be computed to get

$$\lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So we have two independent homogeneous solutions $\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2$ and can form the fundamental matrix

$$F(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & -e^{2t} \end{bmatrix}$$

Using $\det F(t) = -e^{3t}$ we compute (see appendix D for the formula for the inverse of a 2×2 matrix)

$$F(t)^{-1} = -e^{-3t} \begin{bmatrix} -e^{2t} & -e^{2t} \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-t} \\ 0 & -e^{-2t} \end{bmatrix}$$

Guessing $\mathbf{x}_p(t) = F(t)\mathbf{v}(t)$ leads as above to the equation

$$\mathbf{v}(t) = \int_0^t F(\tau)^{-1} \mathbf{r}(\tau) d\tau = \int_0^t \begin{bmatrix} e^{-\tau} & e^{-\tau} \\ 0 & -e^{-2\tau} \end{bmatrix} \begin{bmatrix} e^\tau \\ 0 \end{bmatrix} d\tau = \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Then

$$\mathbf{x}_p(t) = F(t)\mathbf{v}(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & -e^{2t} \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} te^t \\ 0 \end{bmatrix}$$

□

Finally, using fundamental matrices and variation of parameters, we can express the complete solution to an inhomogeneous initial value problem as an integral. That is, suppose we want to solve

$$\mathbf{x}' = A\mathbf{x} + \mathbf{r}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

The general solution can be written (see section 4.2 for explanation of the $\mathbf{x}_h(t)$ term)

$$\mathbf{x}_g(t) = \underbrace{c_1 \mathbf{x}_1(t) + \cdots + c_n \mathbf{x}_n(t)}_{\mathbf{x}_h} + \underbrace{\mathbf{x}_p(t)}_{\mathbf{x}_h(t)} = \underbrace{F(t)\mathbf{c}}_{\mathbf{x}_h(t)} + F(t) \int_0^t F(\tau)^{-1} \mathbf{r}(\tau) d\tau$$

Here it is convenient to set the lower limit of integration to be 0; we'll see why in a second. Solving for \mathbf{c} which satisfies the initial condition as in 4.2, we compute

$$\mathbf{x}_0 = \mathbf{x}(0) = F(0)\mathbf{c} + F(0) \int_0^0 F(\tau)^{-1} \mathbf{r}(\tau) d\tau = F(0)\mathbf{c} \implies \mathbf{c} = F(0)^{-1} \mathbf{x}_0$$

and so the complete solution is

$$\mathbf{x}(t) = F(t)F(0)^{-1}\mathbf{x}_0 + F(t) \int_0^t F(\tau)^{-1}\mathbf{r}(\tau) d\tau$$

or, using the matrix exponential e^{tA} for our $F(t)$,

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 + e^{tA} \int_0^t e^{-\tau A}\mathbf{r}(\tau) d\tau$$

Note the similarity with the integral formula you get for a single first order linear equation using integrating factors as in section 2.2.

4.5 Phase Diagrams for Linear Systems

2×2 linear systems

$$\mathbf{x}' = A\mathbf{x}$$

can be classified into 5 types depending on the behavior of solution curves in the x - y plane. These are

Type	Stability	Eigenvalues
Nodal source	Unstable	Both real, positive
Nodal sink	Stable	Both real, negative
Saddle	Unstable	Both real, opposite signs
Spiral source	Unstable	Complex, positive real part
Spiral sink	Stable	Complex, negative real part

A system is **stable** if all of its solutions approach the origin as $t \rightarrow \infty$. A picture of each type can be seen in figure 3, and examples are discussed below.

Example. Determine the type and sketch the phase diagram for

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}$$

Solution. We examined this equation in section 4.2, and found eigenvalues/eigenvectors

$$\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This system is a **saddle**, and let us see why. Vector \mathbf{v}_1 lies along the line $y = -3/4x$ in the plane. Any solution to the equation which starts on this line will remain on it, since it will be given by the equation (the initial condition will force $c_2 = 0$)

$$\mathbf{x}_1(t) = c_1 e^{-t} \mathbf{v}_1, \text{ for some } c_1$$

So solutions along this line have exponentially *decreasing* magnitude from the e^{-t} term, and approach the origin.

On the other hand, \mathbf{v}_2 lies along the x -axis, and any solution starting on this axis will remain on it, since it will be given by (initial conditions forcing $c_1 = 0$ this time)

$$\mathbf{x}_2(t) = c_2 e^{2t} \mathbf{v}_2, \text{ for some } c_2$$

These solutions have exponentially *increasing* magnitude, and move away from the origin as $t \rightarrow \infty$. For this reason the saddle is unstable.

This is the saddle depicted in figure 3.c. □

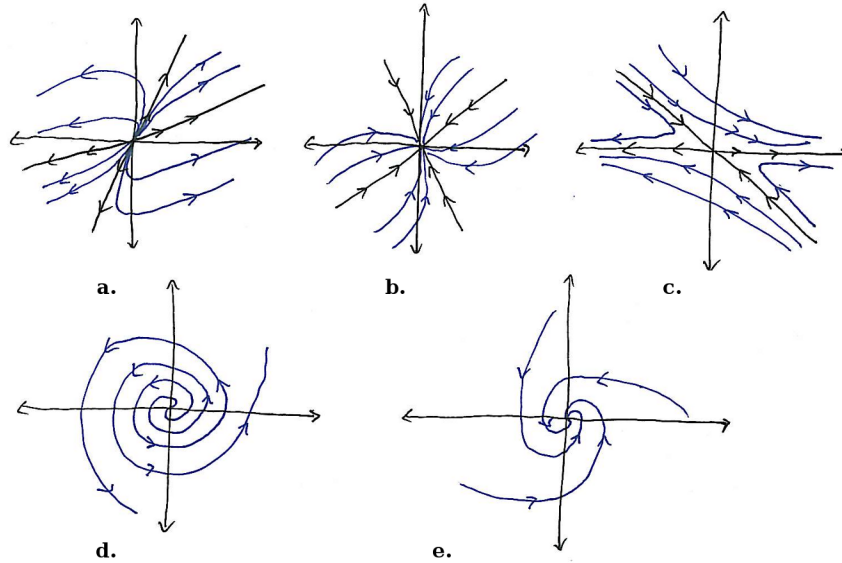


Figure 3: Phase diagrams for a. Nodal source, b. Nodal sink, c. Saddle, d. Spiral source, e. Spiral sink

Example. Determine the type and sketch the phase diagram for

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$$

Solution. We examined this equation in section 4.2, and found that it had complex eigenvalues

$$\lambda = 2 \pm 2i$$

and a general solution

$$\begin{aligned} \mathbf{x}_1 &= \operatorname{Re} \left(e^{2t} (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} \cos 2t + 2 \sin 2t \\ -\cos 2t \end{bmatrix} \\ \mathbf{x}_2 &= \operatorname{Im} \left(e^{2t} (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} \sin 2t - 2 \cos 2t \\ -\sin 2t \end{bmatrix} \\ \mathbf{x} &= c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \end{aligned}$$

Note that the coefficient in the exponent comes from the real part of the eigenvalues $\operatorname{Re}(2 \pm 2i) = 2$. The magnitude of these solutions grows exponentially because of the e^{2t} term, so this is a **spiral source**, with solutions going away from the origin. For this reason it is unstable. The oscillatory functions contribute the spiraling behavior; to determine whether the spiral is clockwise or counterclockwise, we can simply check the following. Suppose a solution curve passes through the point $(1, 0)$ at time $t = 0$, so

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then its *velocity* at time $t = 0$ is given by the differential equation:

$$\mathbf{x}'(0) = A\mathbf{x}(0) = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

this velocity vector points up and to the right, so we conclude that our solutions are spiraling *counterclockwise*. Some solution curves are sketched in figure 3.d. \square

Example. Determine the type and sketch the phase diagram for

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 17 & -3 \\ 3 & 7 \end{bmatrix}$$

Solution. First we determine the eigenvalues and eigenvectors.

$$\det(A - \lambda I) = \det \begin{vmatrix} 17 - \lambda & -3 \\ 3 & 7 - \lambda \end{vmatrix} = \lambda^2 - 24\lambda + 128 = (\lambda - 8)(\lambda - 16) = 0$$

so we have two real, positive eigenvalues. For $\lambda_1 = 8$, we get the eigenvector

$$(A - 8I)\mathbf{v}_1 = \begin{bmatrix} 9 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and for $\lambda_2 = 16$, we get

$$(A - 16I)\mathbf{v}_2 = \begin{bmatrix} 1 & -3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Solutions starting along the line $y = 3x$ through the vector \mathbf{v}_1 will remain on it, with magnitude growing at a rate e^{8t} and hence travel away from the origin. Similarly, solutions starting on the line $y = 1/3x$ through the vector \mathbf{v}_2 will remain on the line, with magnitude growing at a rate e^{16t} and hence travel away from the origin. This system is therefore a **nodal source**, and is unstable.

To examine the behavior of solutions not on either of the eigenvector lines, note that such solutions will be given by a linear combination

$$\mathbf{x}(t) = c_1 e^{8t} \mathbf{v}_1 + c_2 e^{16t} \mathbf{v}_2$$

as $t \rightarrow \infty$, the second term clearly dominates, and so solutions will asymptotically approach the direction of \mathbf{v}_2 for large t . As $t \rightarrow -\infty$, the first term dominates, and so solutions asymptotically approach the direction of \mathbf{v}_1 near the origin (since $t \rightarrow -\infty$). The phase diagram is depicted in figure 3.a. \square

4.6 Nonlinear Systems

A nonlinear (autonomous) system of two equations has the general form

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

We'd like to get a qualitative picture of how various solutions behave, and we can do so in analogy with the phase diagrams of single equations in section 2.3.2. Near a **critical point** (x_0, y_0) such that $f(x_0, y_0) = g(x_0, y_0) = 0$, we can approximate the system by the linear system

$$\begin{aligned} (x - x_0)' &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ (y - y_0)' &= g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) \end{aligned}$$

which follows by expanding f and g in Taylor series around the point (x_0, y_0) and throwing out any terms which are quadratic or higher in the variables $(x - x_0)$, $(y - y_0)$. We can write this in matrix notation as

$$\mathbf{x}' = J(x_0, y_0)\mathbf{x}$$

where $J(x, y)$ is the **Jacobian matrix**

$$J(x, y) = \begin{bmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{bmatrix}$$

We can classify this linear system as one of the 5 types discussed in section 4.5, and sketch the corresponding phase portrait near the critical point. Once we have done this for all of the critical points, we have a good idea of what global solution curves look like.

To summarize, we develop a picture of solution curves in the x - y plane as follows

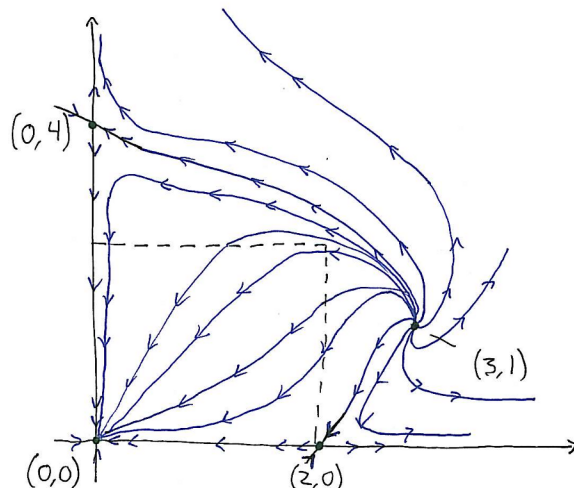


Figure 4: Phase diagram for the nonlinear example

1. Find the critical points, which are points (x_0, y_0) such that $f(x_0, y_0) = g(x_0, y_0) = 0$.
2. Compute the Jacobian $J(x_0, y_0)$ at each critical point.
3. Classify each critical point according to the behavior of the linearized system

$$\mathbf{x}' = J(x_0, y_0)\mathbf{x}$$

4. Analyze long term behavior (showing the solutions are constrained in a particular region, etc.)

Example. Examine the behavior of the nonlinear system

$$\begin{aligned} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{aligned}$$

which models a population $x(t)$ of a prey species, and $y(t)$ of a predator species. What are the possible eventual outcomes of the populations as $t \rightarrow \infty$? What will be the outcome for any set of initial conditions $0 < x(0) < 2$ and $0 < y(0) < 2$?

Solution. First we find the critical points

$$\begin{aligned} 0 &= x(x - 2 - y) \\ 0 &= y(y - 4 + x) \end{aligned}$$

We have 4 possibilities:

$$(x_0, y_0) \in \{(0, 0), (0, 4), (2, 0), (3, 1)\}$$

The last point $(3, 1)$ comes from solving the linear system

$$\begin{cases} 0 = (x - 2 - y) \\ 0 = (y - 4 + x) \end{cases} \iff \begin{cases} x - y = 2 \\ x + y = 4 \end{cases}$$

The Jacobian matrix for this system is

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2x - 2 - y & -x \\ y & 2y - 4 + x \end{bmatrix}$$

- At the critical point $(0, 0)$, we have

$$J(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

which has eigenvalues and eigenvectors

$$\lambda_1 = -2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = -4, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So is a *nodal sink*. (Note that for a source or sink, we don't care too much about the eigendirections, just that all solutions near the critical point are coming in).

- At $(0, 4)$, we have

$$J(0, 4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix} \quad \lambda_1 = -6, \lambda_2 = 4$$

This has two real eigenvalues with opposite signs, so it is a *saddle*. In this case, we do care about the eigendirections, so we compute

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So solutions are *approaching* this critical point along the line $y = -(2/5)x + 4$ (because of the negative eigenvalue), and *departing* from it along the y -axis (corresponding to the positive eigenvalue).

- At $(2, 0)$, we have

$$J(2, 0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix} \quad \lambda_1 = 2, \lambda_2 = -2$$

which is also a saddle. We compute the eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So solutions are *approaching* along $y = 2(x - 2)$ and *departing* along the x -axis.

- At $(3, 1)$, we have (using the formula from appendix D.1 that eigenvalues satisfy $\lambda^2 - \text{tr } A\lambda + \det A = 0$)

$$J(3, 1) = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}, \quad \lambda = \frac{1}{2} \left(\text{tr } J \pm \sqrt{(\text{tr } J)^2 - 4\det J} \right) = \frac{1}{2} (4 \pm \sqrt{16 - 24}) = 2 \pm \sqrt{2}i$$

So this is a *spiral source* (because of the positive real part). To determine its direction we compute (see the complex eigenvalue example in section 4.5 for an explanation of this)

$$\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

so the spiral is *counterclockwise*.

The phase diagram is sketched in figure 4

The possible outcomes appear to be

1. $x(t) \rightarrow 0, y(t) \rightarrow 0$
2. $x(t) \rightarrow \infty, y(t) \rightarrow 0$
3. $x(t) \rightarrow 0, y(t) \rightarrow \infty$, and possibly
4. $x(t) \rightarrow \infty, y(t) \rightarrow \infty$

It appears that all solutions inside the box $0 < x < 2$, $0 < y < 2$ approach $(0, 0)$ as $t \rightarrow \infty$. We can verify this as follows. Provided the vector field $f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is always pointing *into* or *along* the box along the boundary of the box, we can conclude that no solution will ever exit the box.

- Along the boundary $y = 0$, $0 < x < 2$, we have

$$\begin{bmatrix} f(x, 0) \\ g(x, 0) \end{bmatrix} = \begin{bmatrix} x^2 - 2x \\ 0 \end{bmatrix}$$

is a vector whose y component is 0 and whose x component is *negative*, since $x < 2 \implies x^2 - 2x = x(x - 2) < 0$.

- Along the boundary $x = 0$, $0 < y < 2$, we have

$$\begin{bmatrix} f(0, y) \\ g(0, y) \end{bmatrix} = \begin{bmatrix} 0 \\ y^2 - 4y \end{bmatrix}$$

points in the negative y direction since $y < 2 \implies y^2 - 4y = y(y - 4) < 0$.

- Along the boundary $x = 2$, $0 < y < 2$, we have

$$\begin{bmatrix} f(2, y) \\ g(2, y) \end{bmatrix} = \begin{bmatrix} -2y \\ y^2 - 2y \end{bmatrix}$$

the x component of this vector field is negative, so it points *into* the box.

- Along the boundary $y = 2$, $0 < x < 2$, we have

$$\begin{bmatrix} f(x, 2) \\ g(x, 2) \end{bmatrix} = \begin{bmatrix} x^2 - 4x \\ -4 + 2x \end{bmatrix}$$

the y component of this vector field is negative, since $0 < x < 2 \implies -4 + 2x < 0$.

We conclude that all solutions starting in the box must stay in the box. Moreover, the only two critical points which could be approached by solutions in the box are $(0, 0)$ and $(2, 0)$. However, the axis along which solutions approach $(2, 0)$ is $y = 2(x - 2)$ which does not intersect the box, so in fact all solutions in this box approach the final value $(0, 0)$. \square

A Complex Variables

A **complex number** is a number z which can be written in the **Cartesian form**

$$z = x + iy, \quad \text{where } x, y \text{ are real numbers.}$$

and $i^2 = -1$. Then we define the **real part** $\text{Re}(z) = x$ and **imaginary part** $\text{Im}(z) = y$ to be the parts *not multiplying i* and multiplying i , respectively. We can also write z in **polar form**

$$z = re^{i\tau}$$

where $r \geq 0$ and $0 \leq \tau < 2\pi$. We can convert back and forth using **Euler's formula**,

$$e^{i\tau} = \cos \tau + i \sin \tau$$

To go from Cartesian to polar form, we compute

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tau &= \tan^{-1} \frac{y}{x} \end{aligned}$$

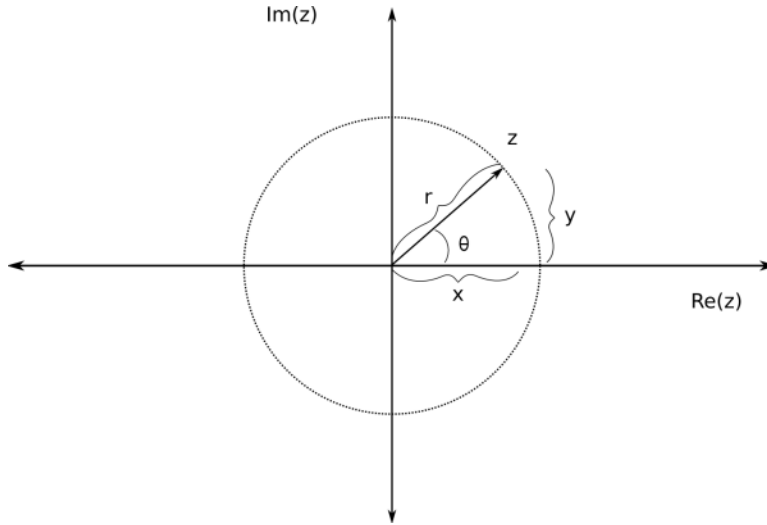


Figure 5: Vectors in the complex plane

and to go from polar to Cartesian we use

$$\begin{aligned}x &= r \cos \tau \\y &= r \sin \tau\end{aligned}$$

See Figure 5.

Also important are the *reverse Euler formulas*

$$\begin{aligned}\cos \tau &= \frac{e^{i\tau} + e^{-i\tau}}{2} \\ \sin \tau &= \frac{e^{i\tau} - e^{-i\tau}}{2i} = \frac{i(e^{-i\tau} - e^{i\tau})}{2}\end{aligned}$$

which are obtained by adding or subtracting $e^{i\tau} = \cos \tau + i \sin \tau$ and $e^{-i\tau} = \cos \tau - i \sin \tau$.

A.1 Multiplication and Division

For two complex numbers z_1 and z_2 , we can compute their product directly, using $i^2 = -1$ by

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + i(x_1 y_2 + x_2 y_1) + i^2 y_2 y_1 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Alternatively, in polar form,

$$z_1 z_2 = (r_1 e^{i\tau_1})(r_2 e^{i\tau_2}) = r_1 r_2 e^{i(\tau_1 + \tau_2)}$$

Division is easy in polar form:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\tau_1}}{r_2 e^{i\tau_2}} = \frac{r_1}{r_2} e^{i\tau_1} e^{-i\tau_2} = \frac{r_1}{r_2} e^{i(\tau_1 - \tau_2)}$$

It is slightly trickier in Cartesian form. We write

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

but then we need to get rid of the i in the denominator. We can always do this by multiplying by the **complex conjugate** of z_2 , denoted by \bar{z}_2 which is defined for any complex number by

$$\bar{z} = x - iy = re^{-i\tau}$$

where we change the sign in front of i . Back in our division problem, we get

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{x_1 + iy_1}{x_2 + iy_2} \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \end{aligned}$$

A.2 Powers and roots

Taking powers of a complex number z^n , is done in the usual manner in Cartesian form, $(x + iy)^n$, but gets harder and harder as n gets large. It is most convenient to do these operations in polar form. Then to raise $z = re^{i\tau}$ to a power, simply compute

$$z^n = (re^{i\tau})^n = r^n e^{in\tau}$$

That is $r \mapsto r^n$ and $\tau \mapsto n\tau$.

Roots are a bit trickier. We do the same thing, replacing n by $1/n$, but we must remember that *there are n different complex roots of any complex number*. To get them all, we note that $e^{i\tau} = e^{i\tau+2\pi i} = e^{i\tau+4\pi i} = \dots$. Thus,

$$z^{1/n} = (re^{i\tau})^{1/n} = \left(re^{i(\tau+2k\pi)}\right)^{1/n} = r^{1/n} e^{i(\tau+2k\pi)/n}, \quad k = 0, 1, 2, \dots, n-1$$

Example. Find the cube roots of i , that is $(i)^{1/3}$.

Solution. First write i in polar form: since $\text{Re}(i) = x = 0$ and $\text{Im}(i) = y = 1$, we get, using the formulas above,

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = 1 \\ \tau &= \tan^{-1} y/x = \tan^{-1} 1 = \pi/2 \end{aligned}$$

The cube roots of i are then given by

$$\begin{aligned} (i)^{1/3} &= \left\{ 1^{1/3} e^{i(\pi/2)/3}, 1^{1/3} e^{i(\pi/2+2\pi)/3}, 1^{1/3} e^{i(\pi/2+4\pi)/3} \right\} \\ &= \left\{ e^{i\pi/6}, e^{i5\pi/6}, e^{i3\pi/2} \right\} \end{aligned}$$

If we want to express these in Cartesian form, we can compute $e^{i\pi/6} = \sqrt{3}/2 + i/2$, $e^{i5\pi/6} = -\sqrt{3}/2 + i/2$ and $e^{i3\pi/2} = -i$, so that

$$(i)^{1/3} = \left\{ \pm\sqrt{3}/2 + i/2, -i \right\}$$

□

A.3 Trigonometric Identities

Most trigonometric identities can be derived quite simply from Euler's formula and the reverse Euler formula, using multiplication of complex exponentials.

$$\cos(\alpha+\beta) = \text{Re} \left(e^{i(\alpha+\beta)} \right) = \text{Re} \left(e^{i\alpha} e^{i\beta} \right) = \text{Re} \left((\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \right) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

and

$$\sin(\alpha+\beta) = \text{Im} \left(e^{i(\alpha+\beta)} \right) = \text{Im} \left(e^{i\alpha} e^{i\beta} \right) = \text{Im} \left((\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \right) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha)$$

The “double” angle, “half” angle identities, and so on are proved similarly.

Using the reverse Euler formula gives a convenient way to express powers of the trig functions $\sin^n \tau$ and $\cos^n \tau$ in terms of $\cos k\tau$ and $\sin k\tau$ for $1 \leq k \leq n$, and vice versa.

Example. Write $\sin^3 \tau$ in terms of $\sin k\tau$, $\cos k\tau$ for $k = 1, 3$.

Solution. By the reverse Euler formula,

$$\begin{aligned} \sin^3 \tau &= \frac{1}{2^3} (i(e^{-i\tau} - e^{i\tau}))^3 = \frac{i^3}{8} (e^{-3i\tau} - 3e^{-i\tau} + 3e^{i\tau} - e^{3i\tau}) \\ &= -\frac{i}{8} [(e^{-3i\tau} - e^{3i\tau}) - 3(e^{-i\tau} - e^{i\tau})] = -\frac{1}{4} \sin 3\tau + \frac{3}{4} \sin \tau \end{aligned}$$

□

Example. Write $\sin 4\tau$ in terms of powers of $\cos \tau$, $\sin \tau$

Solution. We have

$$\sin 4\tau = \text{Im} (e^{4i\tau}) = \text{Im} \left[(e^{i\tau})^4 \right] = \text{Im} \left[(\cos \tau + i \sin \tau)^4 \right]$$

Expanding out using the binomial theorem, and throwing away everything which doesn't multiply i , we get

$$\sin 4\tau = \text{Im} (\cos^4 \tau - 6 \cos^2 \tau \sin^2 \tau + \sin^4 \tau + 4i(\cos^3 \tau \sin \tau - \cos \tau \sin^3 \tau)) = 4 \cos^3 \tau \sin \tau - 4 \cos \tau \sin^3 \tau$$

□

A.4 Sinusoidal Identity

There is a nice trick for writing a linear combination

$$a \cos \omega\tau + b \sin \omega\tau$$

in the form

$$A \cos(\omega\tau - \phi)$$

The trick is to write

$$a \cos \omega\tau + b \sin \omega\tau = \text{Re} ((a - ib)(\cos \omega\tau + i \sin \omega\tau))$$

Converting to polar form, we get $(a - ib) = Ae^{-i\phi}$ where $A = (a^2 + b^2)^{1/2}$ and $-\phi = \tan^{-1} -b/a = -\tan^{-1} b/a$ so $\phi = \tan^{-1} b/a$. That is,

$$\begin{aligned} a \cos \omega\tau + b \sin \omega\tau &= \text{Re} ((a - ib)(\cos \omega\tau + i \sin \omega\tau)) \\ &= \text{Re} (Ae^{-i\phi} e^{i\omega\tau}) \\ &= \text{Re} (Ae^{i(\omega\tau - \phi)}) \\ &= A \cos(\omega\tau - \phi) \end{aligned}$$

$$\text{where } A = (a^2 + b^2)^{1/2}$$

$$\text{and } \phi = \tan^{-1} b/a$$

B Fourier Series

It is a (rather deep) mathematical fact that we can express any periodic function as a series in sines and cosines⁷. A **periodic function** $f(t)$ is one such that

$$f(t) = f(t + P), \quad \text{for all } t$$

and P is the **period** of f ⁸. Setting $L = P/2$ to be the half-period, we can write f by its **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$

where

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(t) dt \\ a_n &= \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt \\ b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt \end{aligned}$$

If $f(t)$ is **even** ($f(t) = f(-t)$), or **odd** ($f(t) = -f(-t)$), we have the following

- If $f(t)$ is even, then $b_n = 0$ for all n and

$$a_n = \frac{2}{L} \int_0^L f(t) \cos \frac{n\pi t}{L} dt$$

- If $f(t)$ is odd, then $a_n = 0$ for all n and

$$b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{L} dt$$

B.1 Periodic Extensions

Given a function $f(t)$ which is defined on an interval $[0, L]$, we can construct various **periodic extensions** $\tilde{f}(t)$, which are periodic functions agreeing with $f(t)$ on $[0, L]$. These are constructed as Fourier series with coefficients computed using $f(t)$. We have

- **the even extension** of period $P = 2L$, $\tilde{f}_{ev}(t) = \tilde{f}_{ev}(-t)$, where

$$\tilde{f}_{ev}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L}, \quad a_n = \frac{2}{L} \int_0^L f(t) \cos \frac{n\pi t}{L} dt$$

- **the odd extension** of period $P = 2L$, $\tilde{f}_{od}(t) = -\tilde{f}_{od}(t)$, where

$$\tilde{f}_{od}(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}, \quad b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{L} dt$$

⁷This can be stated by saying that $\sin \frac{n\pi t}{L}$ and $\cos \frac{n\pi t}{L}$ form an orthonormal basis in the space of periodic functions with period $2L$. The inner product is given by integration over a single period.

⁸There are many periods; for instance, $2P$ and $3P$ are also periods. We call the smallest such P “the” period.

Example. Find the Fourier sine series for the odd, period 4 function $f(t)$, such that $f(t) = t$ on $[0, 2]$.

Solution. The odd extension contains only sine terms, so we compute b_n . In this case $L = 2$ is the half-period.

$$b_n = \int_0^2 t \sin \frac{n\pi t}{2} dt = -\frac{2t}{n\pi} \cos \frac{n\pi t}{2} \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi t}{2} dt = -\frac{4}{n\pi} \cos n\pi + \frac{2}{n\pi} \left[\frac{2}{n\pi} \sin \frac{n\pi t}{2} \right]_0^2$$

But $\sin \frac{n\pi t}{2}$ vanishes at both endpoints $t = 0$ and $t = 2$, and $\cos n\pi = (-1)^n$. Thus we have

$$f(t) = \sum_n b_n \sin \frac{n\pi t}{2}$$

where

$$b_n = \frac{4(-1)^{n+1}}{n\pi}$$

□

B.2 Operations on Series

Given a function $f(t)$ and its Fourier series, we can perform various operations on it to obtain the Fourier series of related functions. For instance, we can get the series for the following transformations of f :

- $f(t) \mapsto cf(t)$ by multiplying each term in the series by c .
- $f(t) \mapsto f(ct)$ by replacing t by ct in each term
- $f(t) \mapsto f(t \pm L)$ by changing the sign of each term with n odd. This is because

$$\cos \frac{n\pi(t \pm L)}{L} = \cos \left(\frac{n\pi t}{L} \pm n\pi \right) = \begin{cases} -\cos \frac{n\pi t}{L} & n \text{ odd} \\ \cos \frac{n\pi t}{L} & n \text{ even} \end{cases}$$

and similarly for sines.

- $f(t) \mapsto f(t - L/2)$ by using the formulas

$$\cos \frac{n\pi(t - (L/2))}{L} = \cos \left(\frac{n\pi t}{L} - n\frac{\pi}{2} \right) = \begin{cases} \cos \frac{n\pi t}{L} & n = 0, 4, 8, \dots \\ -\cos \frac{n\pi t}{L} & n = 2, 6, 10, \dots \\ \sin \frac{n\pi t}{L} & n = 1, 5, 9, \dots \\ -\sin \frac{n\pi t}{L} & n = 3, 7, 11, \dots \end{cases}$$

and similarly for sines.

- $f(t) \mapsto \int_0^t f(t') dt'$ by integrating each term. The constant term (if any) will have to be computed explicitly (see example below).
- $f(t) \mapsto f'(t)$ by taking the derivative of each term. Note that this is only valid if the original series has no points of discontinuity, e.g. if $f(t)$ is everywhere continuous.

Example. Find the Fourier cosine series for the even extension of $\frac{1}{2}t^2$ on $[0, 2]$ using the example above.

Solution. Since $\int_0^t t' dt' = t^2/2$, we will integrate the sine series

$$\sum_n \frac{4(-1)^{n+1}}{n\pi} \sin \frac{n\pi t}{2}$$

term-wise to get our result. We compute

$$\int_0^t b_n \sin \frac{n\pi t'}{2} dt' = -b_n \frac{2}{n\pi} \cos \frac{n\pi t'}{2} \Big|_{t'=0}^t = b_n \frac{2}{n\pi} - b_n \frac{2}{n\pi} \cos \frac{n\pi t}{2}$$

Integrating a sine series therefore gives a cosine series with a_n given by

$$a_n = -b_n \frac{2}{n\pi} = \frac{8(-1)^n}{n^2\pi^2}$$

Collecting the constant terms, we have

$$a_0 = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n^2\pi^2}$$

which isn't very explicit. We can compute a_0 directly, using the definition⁹

$$a_0 = \frac{1}{2} \int_{-2}^2 \frac{t^2}{2} dt = \frac{1}{2} \int_0^2 t^2 dt = \frac{2^3}{6} = \frac{4}{3}$$

In any case, the desired cosine series for the even extension of t^2 on $[0, 2]$ is

$$\frac{a_0}{2} + \sum_n a_n \cos \frac{n\pi t}{2}, \quad a_0 = \frac{4}{3}, \quad a_n = \frac{8(-1)^n}{n^2\pi^2}$$

□

C Laplace Transform

For a function $f(t)$, the Laplace transform $\mathcal{L}(f) = F(s)$ is defined by

$$\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Note that \mathcal{L} is *linear*, that is

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

You should think of \mathcal{L} as a map which takes a function $f(t)$ of t as its *input*, and gives a function $F(s)$ of s as its *output*. The Laplace transform takes most of the basic functions of t , such as t^n , $\cos t$, $\sin t$, $e^{\alpha t}$ and transforms them into *rational* functions of s , that is

$$F(s) \stackrel{\text{typically}}{=} \frac{q(s)}{p(s)}, \quad q(s), p(s) \text{ polynomials in } s$$

Furthermore, it transforms the operation of *differentiation* in t into *multiplication* in s , thus transforming *differential* equations in t into *algebraic* ones in s . A table of Laplace transforms follows

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t) dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$e^{at} f(t)$	$F(s - a)$
$tf(t)$	$-F'(s)$
$u(t - a)f(t - a)$	$e^{-as}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f(t) * g(t)$	$F(s)G(s)$

⁹As an interesting corollary to our computation therefore, we find the value of the series $\sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n^2\pi^2} = \frac{4}{3}$

Example. Verify the identities

$$\begin{aligned}\mathcal{L}(t^k) &= \frac{k!}{s^{k+1}} \\ \mathcal{L}(e^{\alpha t}g(t)) &= G(s - \alpha) \\ \mathcal{L}(\sin \beta t) &= \frac{1}{s^2 + \beta^2}\end{aligned}$$

Solution. Integrating by parts, we have

$$\mathcal{L}(t^k) = \int_0^\infty e^{-st}t^k dt = -\frac{t^k e^{-st}}{s} \Big|_0^\infty + \frac{k}{s} \int_0^\infty t^{k-1} e^{-st} dt = 0 + \frac{k}{s} \mathcal{L}(t^{k-1})$$

Repeating, we have $\mathcal{L}(t^k) = \frac{k!}{s^k} \mathcal{L}(1)$, so now we need only compute $\mathcal{L}(1)$:

$$\mathcal{L}(1) = \int_0^\infty e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^\infty = \frac{1}{s}$$

So

$$\mathcal{L}(t^k) = \frac{k!}{s^{k+1}}$$

Now assume given a function $g(t)$ with Laplace transform $G(s)$. We have

$$\mathcal{L}(e^{\alpha t}g(t)) = \int_0^\infty e^{-st}e^{\alpha t}g(t) dt = \int_0^\infty e^{-(s-\alpha)t}g(t) dt = G(s - \alpha)$$

In particular since $\mathcal{L}(1) = 1/s$ ($t^0 = 1$), we have

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s - \alpha}$$

Finally, we can use the reverse Euler identity

$$\sin \beta t = \frac{e^{i\beta t} - e^{-i\beta t}}{2i}$$

to get

$$\begin{aligned}\mathcal{L}(\sin \beta t) &= \frac{1}{2i} (\mathcal{L}(e^{i\beta t}) - \mathcal{L}(e^{-i\beta t})) \\ &= \frac{1}{2i} \left(\frac{1}{s - i\beta} - \frac{1}{s + i\beta} \right) \\ &= \frac{1}{2i} \left(\frac{(s + i\beta) - (s - i\beta)}{s^2 + \beta^2} \right) \\ &= \frac{\beta}{s^2 + \beta^2}\end{aligned}$$

□

To compute the inverse Laplace transform of a function $F(s)$, we typically use partial fraction decomposition to split $F(s)$ up into a sum of terms whose inverse Laplace transform is obvious by examining the table.

In the examples that follow, we will use the **Heaviside cover up method** for partial fraction decomposition, which allows us to compute the coefficients of some of the partial fractions quickly. In particular, it applies to linear factors only, and in the case of repeated linear factors, only to the one with the highest degree. See the examples for details.

Example. Find $\mathcal{L}^{-1}\left(\frac{1}{(s^2+4)(s-1)}\right)$.

Solution. The denominator is already factored into the irreducible factors $(s^2 + 4)$ and $(s - 1)$. Therefore we want A , B and C such that

$$\frac{1}{(s^2 + 4)(s - 1)} = \frac{As + B}{s^2 + 4} + \frac{C}{s - 1}$$

The Heaviside cover up method is applicable to find C , since $(s - 1)$ is a linear factor, and no higher powers of $(s - 1)$ appear in the denominator of any terms. We “cover up” the $(s - 1)$ in the denominator, and evaluate at $s = 1$. That is,

$$C = \frac{1}{(s^2 + 4)} \Big|_{s=1} = \frac{1}{5}$$

We then proceed as usual to find A and B .

$$\frac{As + B}{s^2 + 4} + \frac{1/5}{s - 1} = \frac{(As + B)(s - 1) + 1/5(s^2 + 4)}{(s^2 + 4)(s - 1)} = \frac{1}{(s^2 + 4)(s - 1)}$$

Looking at coefficients of s^2 in the numerator, we find

$$(A + 1/5)s^2 = 0s^2$$

So $A = -1/5$, and examining coefficients of s ,

$$(B - A)s = 0s$$

So $B = A = -1/5$. Thus,

$$\frac{1}{(s^2 + 4)(s - 1)} = \frac{1}{5} \left(-\frac{s + 1}{s^2 + 4} + \frac{1}{s - 1} \right)$$

We know $\mathcal{L}(e^t) = \frac{1}{s-1}$, but the other term needs a little more nudging. We write

$$\frac{s + 1}{s^2 + 4} = \frac{s}{s^2 + 4} + \frac{1}{2} \frac{2}{s^2 + 4}$$

Then it is obvious by looking at the table above that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2 + 4)(s - 1)}\right) = \frac{1}{5} \left(-\cos 2t - \frac{1}{2} \sin 2t + e^t \right)$$

□

Example. Find $\mathcal{L}^{-1}\left(\frac{s+1}{s^4-2s^3+s^2}\right)$.

Solution. Examining the denominator, we first notice that we can pull out a factor of s^2 :

$$s^4 - 2s^3 + s^2 = s^2(s^2 - 2s + 1)$$

and since $s^2 - 2s + 1$ factors as $(s - 1)^2$, we have the rational function

$$\frac{s + 1}{s^2(s - 1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 1} + \frac{D}{(s - 1)^2}$$

The Heaviside cover up method applies to B and D , since the denominators of those terms contain the highest powers of their respective linear factors. We have

$$B = \frac{s + 1}{(s - 1)^2} \Big|_{s=0} = 1$$

and

$$D = \left. \frac{s+1}{s^2} \right|_{s=1} = 2$$

Then

$$\frac{A}{s} + \frac{1}{s^2} + \frac{C}{s-1} + \frac{2}{(s-1)^2} = \frac{As(s-1)^2 + (s-1)^2 + Cs^2(s-1) + 2s^2}{s^2(s-1)^2}$$

and the numerator of this must be equal to $s+1$. Expanding the numerator, we have

$$As(s-1)^2 + (s-1)^2 + Cs^2(s-1) + 2s^2 = (A+C)s^3 + (3-2A-C)s^2 + (A-2)s + 1 = s+1$$

Comparing terms, we see that $(A+C) = 0$ and $(A-2) = 1$, so that $A = 3$ and $C = -3$. Thus,

$$\frac{s+1}{s^2(s-1)^2} = \frac{3}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{2}{(s-1)^2}$$

To get the inverse Laplace transform of the last term, note the identity $\mathcal{L}(e^{\alpha t}g(t)) = G(s-\alpha)$ from the table. We then write

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)^2}\right) = 3 + t - 3e^t + 2te^t$$

since $\mathcal{L}(te^t) = \frac{1}{(s-1)^2}$. □

Example. Find $\mathcal{L}^{-1}\left(\frac{(s+2)}{s^2+4s+5}\right)$.

Solution. Complete the square in the denominator to get

$$s^2 + 4s + 5 = (s+2)^2 - 4 + 5 = (s+2)^2 + 1$$

So we have the function

$$\frac{(s+2)}{(s+2)^2 + 1}$$

Remember that $\mathcal{L}(e^{\alpha t}g(t)) = G(s-\alpha)$. The above is just the Laplace transform for cosine, but shifted by 2. Indeed we have

$$\mathcal{L}(e^{-2t} \cos t) = \frac{(s+2)}{(s+2)^2 + 1}$$

so

$$\mathcal{L}^{-1}\left(\frac{(s+2)}{s^2+4s+5}\right) = e^{-2t} \cos t$$
□

C.1 Convolution and the Laplace Transform

Given two functions $f(t)$ and $g(t)$, we can form their **convolution product**, which is a new function of t given by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau) d\tau$$

This product is both *associative*, $f * (g * h) = (f * g) * h$; and *commutative*, $f * g = g * f$. The Laplace transform takes convolution in t into ordinary multiplication in s :

$$\mathcal{L}(f(t) * g(t)) = F(s)G(s)$$

This can sometimes be useful in computing otherwise difficult convolutions.

Example. Compute

$$t^m * t^n$$

Solution. The integral

$$t^m * t^n = \int_0^t \tau^m (t - \tau)^n d\tau$$

is somewhat difficult to compute directly, involving m integration by parts. Using the Laplace transform on the other hand, we compute

$$\mathcal{L}(t^m * t^n) = \left(\frac{m!}{s^{m+1}}\right) \left(\frac{n!}{s^{n+1}}\right) = \frac{m!n!}{s^{m+n+2}} = \frac{m!n!}{(m+n+1)!} \left(\frac{(m+n+1)!}{s^{m+n+2}}\right)$$

so

$$t^m * t^n = \mathcal{L}^{-1} \left(\frac{m!n!}{(m+n+1)!} \left(\frac{(m+n+1)!}{s^{m+n+2}} \right) \right) = \frac{m!n!}{(m+n+1)!} t^{m+n+1}$$

□

C.2 Piecewise Functions & Delta Functions

The Laplace transform is often used in conjunction with **piecewise functions** which are functions of the form

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1 \\ f_2(t) & t_1 < t < t_2 \\ \vdots & \vdots \end{cases}$$

To express piecewise functions on one line, we typically use the **unit step function** defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

We can easily create a “box” function which is equal to 1 on an interval $[a, b]$ and equal to 0 elsewhere by writing

$$f(t) = u(t - a) - u(t - b) = \begin{cases} 0 & t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases}$$

Think of the first term as a switch which “turns on” a 1 at time $t = a$, and the second term as a switch which turns on a -1 at time $t = b$ and so cancels out the first term.

Using such box functions, we can write

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1 \\ f_2(t) & t_1 < t < t_2 \\ \vdots & \vdots \end{cases}$$

as

$$f(t) = (u(t - t_0) - u(t - t_1))f_1(t) + (u(t - t_1) - u(t - t_2))f_2(t) + \dots$$

This is in a more useful form for taking the Laplace transform, for instance (see example in section 3.3)

When we take derivatives of piecewise functions, we often obtain **spikes**. The primordial jump is the **Dirac delta function** $\delta(t)$ which is not really a function but has the following properties

$$\delta(t) = 0, \quad \text{for } t \neq 0, \quad \int_{-\infty}^{\infty} \delta(t) dt = \int_{-\epsilon}^{\epsilon} \delta(t) dt = 1$$

so it has a finite amount of area under the “curve” at the point $t = 0$. More generally, $\delta(t)$ satisfies

$$\int_{-\infty}^{\infty} \delta(t - a)f(t) dt = f(a)$$

that is, if we integrate a function $f(t)$ against a delta function living at $t = a$, we get the value of the function at a .

The unit step function has a jump of height 1 at $t = 0$, and its derivative $u'(t)$ has a *spike* of total area 1 at $t = 0$, namely

$$u'(t) = \delta(t)$$

This is true more generally. Any time a piecewise function has a jump of height h (negative if the jump is “down”), its **generalized derivative**¹⁰ will have a spike of area h given by a term $h\delta(t)$.

Example. Let $f(t)$ be given by

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 2 \\ 1 & 2 < t < 3 \\ 0 & t > 3 \end{cases}$$

Write $f(t)$ in terms of step functions and find its generalized derivative $f'(t)$.

Solution. To write $f(t)$ in terms of step functions, we think of creating boxes defined on $[0, 2]$ and $[2, 3]$. Then

$$f(t) = (u(t) - u(t - 2))t + (u(t - 2) - u(t - 3))$$

Note that $f(t)$ has two jumps of height -1 (since we jump down as we go from left to right) at $t = 2$ and at $t = 3$.

It’s generalized derivative is most easily thought of as follows. On the regions where $f'(t)$ is well-defined, we have

$$f'(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 2 \\ 0 & 2 < t < 3 \\ 0 & t > 3 \end{cases}$$

But we also have to include the spikes at $t = 2$ and $t = 3$ of area -1 . We obtain

$$f'(t) = (u(t) - u(t - 2)) - \delta(t - 2) - \delta(t - 3)$$

□

D Matrices

Here we recall some facts about matrices. Recall that matrices can be multiplied together (AB makes sense if the number of columns of A equals the number of rows of B), to get a multiplication which is *associative*: $A(BC) = (AB)C$, but not necessarily commutative: $AB \neq BA$ in general. The unit element is the **identity matrix**

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

meaning that $AI = IA = A$ for all matrices A for which either product makes sense.

¹⁰We call this the generalized derivative since the function doesn’t technically have a derivative in the usual sense. Any time you differentiate something to get something with jumps or spikes, you’re using generalized derivatives.

An $n \times n$ matrix A is **invertible** if and only if its **determinant** is nonzero: $\det A \neq 0$. Recall that the determinant of a 2×2 matrix is given by

$$\det A = \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 matrix, you can evaluate its determinant by expanding along a row or column. For each entry you have a term equal to that entry times the determinant of the 2×2 matrix obtained by deleting the corresponding row and column. Then you add the three terms up, with an alternating sign (+1 starting in the upper left, and changing each time you step right or down). For example,

$$\begin{aligned} \det A = \det \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge) \end{aligned}$$

So provided $\det A \neq 0$, there is a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

For 2×2 matrices, there is a handy formula for the inverse of a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where the diagonal entries switch places, the off diagonal entries get opposite signs, and we multiply the whole thing by the scalar quantity $1/\det A$.

D.1 Eigenvalues & Eigenvectors

Given a matrix A , it frequently happens that we are interested in finding scalars λ and vectors \mathbf{v} such that the action of A on \mathbf{v} is just the same as multiplying \mathbf{v} by λ .

$$A\mathbf{v} = \lambda\mathbf{v}$$

Solutions to this equation are called **eigenvalues** for the scalars λ , and **eigenvectors** for the vectors \mathbf{v} .

The above equation can be rewritten as $(A - \lambda I)\mathbf{v} = \mathbf{0}$, and of course the only way we can get a nonzero vector \mathbf{v} satisfying this equation is if $(A - \lambda I)$ is *not* invertible (otherwise we'd have $\mathbf{v} = (A - \lambda I)^{-1}\mathbf{0} = \mathbf{0}$). So we determine the possible eigenvalues from the **eigenvalue equation**

$$\det(A - \lambda I) = 0$$

Then, for a given eigenvalue λ , we can try to find an associated eigenvector by solving the **eigenvector equation**

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

In the case of a 2×2 matrix A , the eigenvalue equation has a special form which may be worth memorizing:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies 0 = \det(A - \lambda I) = \lambda^2 - \text{tr } A + \det A, \quad \text{tr } A = (a + d), \quad \det A = (ad - bc)$$

where $\text{tr } A = a + d$ is called the **trace** of A , defined for any matrix as the sum of the diagonal entries. Be careful to note the minus sign in front of the trace in the quadratic polynomial for λ .

Remark. The case of repeated eigenvalues is discussed in section 4.2 and will not be covered here.

Example. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

Solution. The eigenvalue equation is

$$\det(A - \lambda I) = \det \begin{vmatrix} 3 - \lambda & 4 \\ 4 & -3 - \lambda \end{vmatrix} = (3 - \lambda)(-3 - \lambda) + 16 = \lambda^2 - 25 = \lambda^2 - \text{tr } A + \det A = (\lambda + 5)(\lambda - 5) = 0$$

so we have eigenvalues $\lambda = 5$ and $\lambda = -5$.

For $\lambda = 5$, we want to solve

$$(A - 5I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives the equation $-2a + 4b = 0$ (the second equation $4a - 8b = 0$ is equivalent to the first as it must be to get a nonzero solution). We can choose any convenient value for a and b which solves this equation, say $a = 2$, $b = 1$. So

$$\lambda = 5, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is an eigenvalue, eigenvector pair.

Similarly, for $\lambda = -5$ we solve

$$(A - (-5)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So we can take $a = 1$, $b = -2$, for instance and get

$$\lambda = -5, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

for our second pair. □

D.2 Matrix Exponentials

Another important quantity associated to a matrix A is the **matrix exponential**

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n$$

obtained by applying the Taylor series $e^x = \sum_n \frac{x^n}{n!}$ to the matrix quantity tA . The result¹¹ is a matrix which is a function of t , and is important in the theory of homogeneous systems of equations (see section 4.3). It is a solution to the differential equation characterizing fundamental matrices for A (see section 4.3), and is the unique one satisfying the additional initial condition

$$(e^{tA})' = Ae^{tA}, \quad e^{0A} = I$$

We can verify this by examining the series:

$$\begin{aligned} (e^{tA})' &= \frac{d}{dt} \left(I + tA + \frac{t^2}{2!} + \dots \right) = A + \frac{2t}{2!}A^2 + \frac{3t}{3!}A^3 = A \left(I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \dots \right) \\ e^{0A} &= I + 0A + 0A^2 + \dots = I \end{aligned}$$

¹¹It can be shown that the resulting series of matrices converges for all t

In some cases we can evaluate the series directly, for instance when a matrix is **diagonal** (for which entries are zero except along the diagonal), or **strictly upper/lower triangular** (for which the only nonzero entries are either above the main diagonal or below it).

In other cases, we can compute the matrix exponential using fundamental matrices, which is discussed in section 4.3.

Example. Compute e^{tA} and e^{tB} , where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution. A is a diagonal matrix, as defined above. As we compute its successive powers, we see that we end up just taking the corresponding powers of the entries along the diagonal

$$A^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1^3 & 0 \\ 0 & 2^3 \end{bmatrix}, \quad A^n = \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix}, \text{ etc.}$$

Then for the exponential series, we obtain

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix} = \begin{bmatrix} \sum_n \frac{(1t)^n}{n!} & 0 \\ 0 & \sum_n \frac{(2t)^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{1t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

B is a strictly upper triangular matrix. Such matrices have the property (called *nilpotent*) that $B^n = 0$ for all sufficiently large n . In this case,

$$B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, \quad B^3 = BB^2 = 0, \dots, B^n = 0, \quad \text{for } n > 1$$

So in the exponential series for B , we have

$$e^{tB} = I + tB + \frac{t^2}{2!} B^2 + \dots = I + tB = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

□

Finally, we might expect the usual rule for multiplication of exponentials to hold for matrices, that is, $e^{A+B} = e^A e^B$. **This is not true in general**, however. For this to hold, we need A and B to **commute**; that is,

$$e^{A+B} = e^A e^B \quad \text{only if } AB = BA$$

The reason for this can be seen by examining the series:

$$e^{A+B} = I + (A+B) + \frac{1}{2} (A+B)^2 + \dots = I + (A+B) + \frac{1}{2} (A^2 + B^2 + AB + BA) + \dots$$

whereas

$$e^A e^B = \left(I + A + \frac{A^2}{2} + \dots \right) \left(I + B + \frac{B^2}{2} + \dots \right) = I + (A+B) + \frac{1}{2} (A^2 + B^2 + 2AB) + \dots$$

so the two series are only equal (up to second order at least, but a similar thing happens for all orders) if $AB + BA = 2AB \iff AB = BA$.

As a useful corollary to this, note that

$$(e^{tA})^{-1} = e^{-tA}$$

that is, the inverse matrix to e^{tA} is just the matrix exponential itself, evaluated at $-t$. This follows from using the multiplication rule, which applies here since A clearly commutes with itself ($AA = A^2 = AA$), so

$$e^{tA} e^{-tA} = e^{t(A-A)} = e^{t0} = I$$

which is the defining property for the inverse matrix

Example. Compute e^{tA} where

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_B + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_C$$

Solution. A is the sum of a diagonal matrix B and a strictly upper triangular matrix C . Using the techniques from the previous example, we compute

$$e^{tB} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}, \quad e^{tC} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

In order to use the multiplication rule, we need to check commutativity:

$$\begin{aligned} BC &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \\ CB &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Since $BC = CB$, we can compute

$$e^{tA} = e^{t(B+C)} = e^{tB}e^{tC} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

□

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