BUNDLES, CLASSIFYING SPACES AND CHARACTERISTIC CLASSES

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Introduction

Fiber bundles, especially vector bundles, are ubiquitous in mathematics. Given a space B, we would like to classify all vector bundles on B up to isomorphism. While we accomplish this in a certain sense by showing that any vector bundle $E \longrightarrow B$ is isomorphic to the pullback of a 'universal vector bundle' $E' \longrightarrow B'$ (depending on the rank and field of definition) by a map $f : B \longrightarrow B'$ which is unique up to homotopy, in practice it is difficult to compute the homotopy class of this 'classifying map.'

We can nevertheless (functorially) associate some *invariants* to vector bundles on B which may help us to distinguish them. (Compare the problem of classifying spaces up to homeomorphism, and the partial solution of associating functorial

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invariants such as (co)homology and homotopy groups.) These invariants will be cohomology classes on B called *characteristic classes.* In fact all characteristic classes arise as cohomology classes of the universal spaces B'.

1. Bundles

Definition 1.1. A **fiber bundle** is a triple (π, E, B) consisting of a locally trivial, continuous surjection

$$\pi : E \longrightarrow B$$

from the **total space** E to the **base space** B. Here 'locally trivial' means that for all $b \in B$, there is an open neighborhood $U \ni b$ whose preimage $\pi^{-1}(U)$ is homeomorphic to the product of U with a fixed *fiber space* F, in such a way that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

(pr_1 denotes projection onto the first factor.) As a matter of notation, the fiber bundle (π, E, B) is often denoted just by E or π. It follows that for each $b \in B$, the *fiber over* b (which will be denoted by $\pi^{-1}(b)$ or E_b) is homeomorphic to F.

Definition 1.2. A **morphism** of fiber bundles $f : (\pi, E, B) \longrightarrow (\pi', E', B')$ consists of continuous maps $f : E \longrightarrow E'$ and $\widetilde{f} : B \longrightarrow B'$ such that

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "\widetilde{f}"] & B' \end{tikzcd}$$

commutes. We often denote the morphism simply by $f : E \longrightarrow E'$, and say that f is a **morphism over the map** $\widetilde{f} : B \longrightarrow B'$. In particular, observe that f maps fibers to fibers; i.e. for $b \in B$, f restricts to a map $f : F \cong \pi^{-1}(b) \longrightarrow \pi'^{-1}\big(\widetilde{f}(b)\big) \cong F'$.

With these definitions, it is easy to verify that fiber bundles form a category. In addition, we can fix the base space B, and speak of the category of **fiber bundles over** B, where the morphisms are required to lie over the identity map $\mathrm{Id} : B \longrightarrow B$.

Example 1.3. The basic example of a fiber bundle over B with fiber space F is the product $E = B \times F$, with projection onto the first factor. This is known as the **trivial bundle**, and we say that any bundle E is trivial if it is isomorphic to $B \times F$.

1.1. Pullback.

Perhaps the most important way of obtaining new fiber bundles from existing ones is via pullback.

Proposition 1.4. *If $\pi : E \longrightarrow B$ is a fiber bundle with fiber space F and if $f : A \longrightarrow B$ is a continuous map, then the* **pullback**

$$f^*(E) \equiv A \times_B E = \{(a, e) \in A \times E : f(a) = \pi(e)\}$$

is a fiber bundle over A, also with fiber space F, and there is a canonical morphism $\mathrm{pr}_2 : f^*(E) \longrightarrow E$ *lying over f:*

$$\begin{tikzcd} f^*(E) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Furthermore, any morphism $f : (\pi, E, B) \longrightarrow (\pi', E', B')$ of fiber bundles factors as the composition of a morphism $\phi : E \longrightarrow \widetilde{f}^*(E')$ over $\mathrm{Id} : B \longrightarrow B$ followed by the canonical morphism $\mathrm{pr}_2 : \widetilde{f}^*(E') \longrightarrow E'$, where $\widetilde{f} : B \longrightarrow B'$ is map on base spaces induced from f.

Proof. The diagram is an immediate consequence (really the universal defining property) of the pullback operation. To verify that $\mathrm{pr}_1 : f^*(E) \longrightarrow A$ is a fiber bundle, consider a trivialization $h : \pi^{-1}(U) \xrightarrow{\cong} U \times F$ over $U \subset B$ and let $V = f^{-1}(U) \subset A$. The trivialization composed with a projection defines a continuous map

$$\mathrm{pr}_1^{-1}(V) = \big\{(a, e) \in V \times \pi^{-1}(U) : f(a) = \pi(e)\big\} \xrightarrow{1\times h} \\ \{(a, b, z) \in V \times U \times F : f(a) = b\} \longrightarrow \{(a, z)\} = V \times F.$$

This has a continuous inverse given by $(a, z) \longmapsto \big(a, h^{-1}(f(a), z)\big)$.

The second claim follows directly from the universal property of pullback. $\square$

Example 1.5 (Restriction)**.** A special example of pullback is restriction to a subspace, in which $A \subset B$ and f is the inclusion map. In this case it is easily seen that $f^*(E) = \pi^{-1}(A) \subset E$.

1.2. **Sections.** A fiber bundle $\pi : E \longrightarrow B$ is the setting for a special class of maps $B \longrightarrow E$ called *sections.*

Definition 1.6. A **(global) section** of $\pi : E \longrightarrow B$ is a continuous map $s : B \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}$. Thus s maps points $b \in B$ to points in the fibers $\pi^{-1}(b)$.

More generally, for a subspace $U \subset B$, a **(local) section over** U is a map $s : U \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_U$. The set of sections of E over U will be denoted $\Gamma(U, E)$, and we write $\Gamma(E) := \Gamma(B, E)$ for global sections.

A fiber bundle may or may not admit global sections (we'll see this most clearly in the case of principal bundles), but it always admits local sections:

Proposition 1.7. *If $E \longrightarrow B$ has a local trivialization $h : \pi^{-1}(U) \longrightarrow U \times F$ then sections $s \in \Gamma(U, E)$ are in bijection with continuous maps $\widetilde{s} \in \mathrm{Map}(U, F)$.*

Proof. Sections $s : U \longrightarrow \pi^{-1}(U)$ can be composed with h to obtain maps of the form $h \circ s : U \ni b \longmapsto (b, \widetilde{s}(b)) \in U \times F$, and conversely, given $\widetilde{s} : U \longrightarrow F$, $b \longmapsto h^{-1}(b, \widetilde{s}(b))$ is a section. $\square$

1.3. **Fiber bundles as fibrations.** Recall that a **fibration** is a map $p : E \longrightarrow B$ which satisfies the **homotopy lifting property** (HLP) that every homotopy $f : A \times I \longrightarrow B$ and map $\widetilde{f}_0 : A \times 0 \longrightarrow E$ lifts to a homotopy $\widetilde{f} : A \times I \longrightarrow E$ such that $\widetilde{f}|A \times 0 \equiv \widetilde{f}_0$ and $p \circ \widetilde{f} = f$.

It is easy to see that a trivial fiber bundle is a fibration, and hence every fiber bundle is in some sense a 'local fibration,' in that homotopies may be lifted locally on sets over which the bundle is trivial. In fact, with some conditions on the base space, it is possible to show that a local fibration in this sense is indeed a fibration. The proof of the following theorem is rather technical and will be omitted.

Theorem 1.8. *If (π, E, B) is a fiber bundle and B is paracompact, then (π, E, B) is a fibration.*

Remark. In fact all that is needed is the existence of a single locally finite cover of B by open sets over which E is trivial and which admits a subordinate **partition of unity** (the condition of paracompactness implies that *every* open cover has a refinement with this property). Such a cover is called **numerable**, and E is called a 'numerable fiber bundle.' See [Dol63].

From this point on, we shall assume that all base spaces are paracompact, a condition which is satisfied in practice by essentially all spaces of interest, including manifolds and CW complexes, and therefore that fiber bundles are fibrations. In particular, under this assumption it follows that to each fiber bundle (π, E, B) with fiber F we have a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

In practice we are most often interested in categories of fiber bundles where the fiber spaces F are equipped with some algebraic structure, most notably that of a vector space in the case of vector bundles (or so-called 'G-torsors' in the case of principal bundles). This can be formulated by saying that we are choosing some particular class of automorphisms of F and requiring that $\mathrm{Aut}(F)$ to be preserved by the bundle morphisms, trivializations and so on. For instance, a vector space V is a topological space, but we are mostly interested in maps which preserve the linear structure and so we consider $\mathrm{Aut}(V) = \mathrm{GL}(V)$ instead of $\mathrm{Homeo}(V)$. We refer to $\mathrm{Aut}(F)$ as the **structure group** of F. The general theory of fiber bundles with fixed structure group is neatly encapsulated by the machinery of principal bundles. However, we will next talk about vector bundles since these are the objects of primary interest.

2. Vector bundles

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Briefly, a vector bundle is a fiber bundle with structure group $\mathrm{GL}(n, \mathbb{F})$. More precisely,

Definition 2.1. A **vector bundle of rank** n is a fiber bundle (π, E, B) whose fibers $\pi^{-1}(b)$ have the structure of n dimensional vector spaces over $\mathbb{F}$, and whose local trivializations $h_U : \pi^{-1}(U) \cong U \times \mathbb{F}^n$ restrict to linear isomorphisms $h_U : \pi^{-1}(b) \cong \{b\} \times \mathbb{F}^n$. A rank 1 bundle is often referred to as a *line bundle.*

A *morphism of vector bundles* is a fiber bundle morphism which restricts to a linear map on each fiber.

Example 2.2. A smooth n-manifold M has a canonical **tangent bundle** $TM \longrightarrow M$ which is a rank n real vector bundle. If $L \longrightarrow M$ is an embedding of another smooth manifold of dimension l, then there are several canonical vector bundles over L. In addition to the (intrinsic) tangent bundle $TL \longrightarrow L$, there is the restriction of TM to L, and the **normal bundle** $NL = TM/TL \longrightarrow L$. In the last example, the quotient means that at each point $p \in L$, the fiber space is the linear quotient TM_p/TL_p.

Example 2.3. For any space B, we may form the **trivial vector bundle**

$$\underline{\mathbb{F}}^n := B \times \mathbb{F}^n,$$

and as before say that any vector bundle over B isomorphic to $B \times \mathbb{F}^n$ is trivial.

As an exercise, consider the embedding $S^n \subset \mathbb{R}^{n+1}$ as

$$\mathbb{R}^{n+1} \supset S^n \left\{(x_1, \dots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\right\}$$

and show that that normal bundle $NS^n \longrightarrow S^n$ with respect to this embedding is a trivial line bundle.

Example 2.4. Consider the manifold $\mathbb{C}P^n$ consisting of the set of complex lines $\left\{l \subset \mathbb{C}^{n+1}\right\}$. in $\mathbb{C}^{n+1}$. Form the trivial bundle $\underline{\mathbb{C}}^{n+1} \longrightarrow \mathbb{C}P^n$ and consider the subbundle

$$\gamma_n^1 := \left\{(l, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : v \in l\right\} \subset \underline{\mathbb{C}}^{n+1} \longrightarrow \mathbb{C}P^n$$

Equipped with the projection onto the first factor, this forms a complex rank 1 vector bundle over $\mathbb{C}P^n$ called the **canonical complex line bundle.**

The **canonical real line bundle** $\gamma_n^1 \longrightarrow \mathbb{R}P^n$ is defined similarly as

$$\gamma_n^1 := \left\{(l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in l\right\} \subset \underline{\mathbb{R}}^{n+1} \longrightarrow \mathbb{R}P^n$$

2.1. **Whitney sum.**

The pullback construction for fiber bundles specializes to the category of vector bundles; the proof of the following is straightforward and left to the reader.

Proposition 2.5. *If $\pi : E \longrightarrow B$ is a rank n vector bundle and $f : A \longrightarrow B$ a continuous map, then $f^*(E) \longrightarrow A$ is a rank n vector bundle admitting a vector bundle morphism $f^*(E) \longrightarrow E$ over f.*

An important instance of this is the following. First of all, given vector bundles (π_1, E_1, B_1), (π_2, E_2, B_2), the product

$$\pi_1 \times \pi_2 : E_1 \times E_2 \longrightarrow B_1 \times B_2$$

is a vector bundle over $B_1 \times B_2$. If the fibers of E_i are denoted by $V_i \cong \mathbb{F}^{n_i}$, $i = 1, 2$, then it is easily seen that $E_1 \times E_2$ has fibers $V_1 \times V_2 \equiv V_1 \oplus V_2 \cong \mathbb{F}^{n_1+n_2}$. There is an analogous construction in the category of vector bundles over a fixed base B.

Definition 2.6. Given two vector bundles $\pi_i : E_i \longrightarrow B$, $i = 1, 2$ over the same base, the **Whitney sum** is the vector bundle denoted $E_1 \oplus E_2$ which is given by restricting $E_1 \times E_2 \longrightarrow B \times B$ to the diagonal $\mathrm{Diag} : B \subset B \times B$. In other words,

$$E_1 \oplus E_2 := \mathrm{Diag}^*(E_1 \times E_2) \longrightarrow B,$$

where $\mathrm{Diag} : b \longmapsto (b, b)$. The Whitney sum has fibers isomorphic to $\mathbb{F}^{n_1} \oplus \mathbb{F}^{n_2}$ where $n_i = \mathrm{rank}(E_i)$.

2.2. **Sections of vector bundles.** Since the fibers of a vector bundle (π, E, B) have a linear structure, it follows that sections $\Gamma(U, E)$ of E form a vector space. In other words, given two sections $s_1, s_2 \in \Gamma(U, E)$, the linear combinations

$$(a_1 s_1 + a_2 s_2) : b \longmapsto \big(a_1 s_1(b) + a_2 s_2(b)\big) \in E_b, \quad a_i \in \mathbb{F}$$

are again in $\Gamma(U, E)$. Vector bundles always have at least one (global) section, namely the **zero section** z which is given by $z(b) = 0$ for all b. This is well-defined since the point $0 \in \mathbb{F}^n$ is preserved by all linear isomorphisms.

There is a characterization of trivial vector bundles in terms of sections; though this is a direct consequence of Proposition 3.4, we will give a direct proof for vector bundles here.

Proposition 2.7. *A rank n vector bundle $\pi : E \longrightarrow B$ is trivial over $U \subset B$ if and only if there exists a collection of n linearly independent sections (called a* **frame***) $\{s_1, \dots, s_n\} \in \Gamma(U, E)$. Here linear independence means that for each $b \in U$, the set $\{s_1(b), \dots, s_n(b)\}$ is linearly independent. (In particular each section is nowhere vanishing.)*

Proof. If $h : \pi^{-1}(U) \cong U \times \mathbb{F}^n$ is a trivialization, then $s_i : b \longmapsto h^{-1}(b, e_i)$, $i = 1, \dots, n$ give such a collection, where e_i denotes the ith standard basis vector for $\mathbb{F}^n$.

Conversely, given $\{s_1, \dots, s_n\}$, we may form a trivialization $h : U \times \mathbb{F}^n \cong \pi^{-1}(U)$ by

$$h : \Big(b, \sum a_i e_i\Big) \longmapsto \sum a_i s_i(b).$$

Linear independence and dimensional considerations show this to be an isomorphism for each fixed b. $\square$

2.3. **Inner products.** An **inner product** on a vector bundle $E \longrightarrow B$ is a map $\langle\cdot,\cdot\rangle : E \oplus E \longrightarrow \mathbb{F}$ such that $\langle\cdot,\cdot\rangle$ restricts to a symmetric (in case $\mathbb{F} = \mathbb{R}$) or Hermitian (in case $\mathbb{F} = \mathbb{C}$) positive definite bilinear form on each fiber.

Proposition 2.8. *If B is paracompact, then any vector bundle $E \longrightarrow B$ admits an inner product.*

Proof. Paracompactness means that B has a locally finite covering $\{U_\alpha\}$ with a subordinate *partition of unity* $\{\phi_\alpha\}$, meaning that $\phi_\alpha : B \longrightarrow [0, 1]$, $\overline{\phi_\alpha^{-1}(0, 1]} \subset U_\alpha$ and $\sum_\alpha \phi_\alpha \equiv 1$. Refining if necessary, we may assume that E has local trivializations $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{F}^n$.

Over each trivialization we have an inner product by pulling back the standard inner product on $\mathbb{F}^n$ by the h_α, and we may sum these using the partition of unity:

$$\langle v, w\rangle := \sum_\alpha \phi_\alpha(\pi(v)) \, \langle h_\alpha(v), h_\alpha(w)\rangle_{\mathbb{F}^n} .$$

By paracompactness the sum is locally finite, and since inner products are closed under convex combinations, the result follows. $\square$

We will return to vector bundles and their characteristic classes once we have developed the machinery of principal bundles.

3. Principal Bundles

Principal bundles are so named because they give rise via the associated bundle construction below to all fiber bundles, with arbitrary structure groups.

Definition 3.1. Fix a topological group G. A **principal G-bundle** over a space B is a fiber bundle (π, P, B) with a free and transitive right action by G on the fibers.

Recall that a free action is one such that $p \cdot g = p$ iff $g = e$, and transitivity means that for $p, p' \in \pi^{-1}(b)$, there exists a $g \in G$ such that $p \cdot g = p'$ (which is unique in light of freeness).

It follows that P has fiber space homeomorphic to G, and we require that local trivializations $h : \pi^{-1}(U) \cong U \times G$ intertwine the right action with right translation: $h(p) = (u, g) \implies h(p \cdot g') = (u, gg')$.

An equivalent definition is that P is a space with a free right G action such that B is the quotient $\pi : P \longrightarrow P/G = B$, with a corresponding local trivialization condition.

The existence of a unique $g \in G$ relating any two $p, p' \in \pi^{-1}(b)$ defines a **translation function** on the fibers:

$$(p, p') \in P_b^2 \longmapsto \tau(p, p') \in G, \text{ such that } p \cdot \tau(p, p') = p', \tag{1}$$

which will be occasionally useful.

Remark. It is important to note that a principal G-bundle is *different* from a fiber bundle whose fibers are equipped with a group structure isomorphic to G. In particular, there is no canonical identity e in a fiber; rather the fibers of P have the structure of so-called '**G-torsors**,' which are to groups what affine spaces are to vector spaces.

Another difference can be seen by comparing two different local trivializations $h_i : \pi^{-1}(U) \cong U \times G$, $i = 1, 2$. For a fiber bundle with a group structure these would differ by $h_2 \circ h_1^{-1} : (b, g) \longmapsto (b, \phi(b)g)$ where the $\phi(b)$ are group isomomorphisms. For a principal bundle on the other hand, the transition functions $b \longmapsto \phi(b)$ must be right translations by elements of G, and are not even homomorphisms.

3.1. Morphisms.

Definition 3.2. A **morphism of principal G-bundles** $\phi : (\pi_1, P_1, B_1) \longrightarrow (\pi_2, P_2, B_2)$ is a fiber bundle morphism which intertwines the G actions; i.e. $\phi(p \cdot g) = \phi(p) \cdot g$. (We also say ϕ is **equivariant** with respect to the G actions.)

In fact *any* equivariant map $\phi : P_1 \longrightarrow P_2$ is a principal bundle morphism since $\phi(p \cdot g) = \phi(p) \cdot g$ means that ϕ maps fibers to fibers, hence $\widetilde{\phi}(b) := \pi_2\big(\phi(p)\big)$ is well-defined for any choice of $p \in \pi_1^{-1}(b)$ and specifies the base map $\widetilde{\phi} : B_1 \longrightarrow B_2$ uniquely.

It turns out that morphisms of G-bundles over B are tightly constrained:

Proposition 3.3. *Let P and P' be principal G-bundles over B. If $\phi : P \longrightarrow P'$ is a morphism lying over* $\mathrm{Id} : B \longrightarrow B$, *then ϕ is an isomorphism.*

Proof. To see that ϕ is injective, suppose $\phi(p) = \phi(q)$ for two points $p, q \in P$. Since ϕ lies over the identity on B, it follows that p and q must lie in the same fiber $\pi^{-1}(b) \subset P$. Then there is a unique $g = \tau(p, q)$ such that $p \cdot g = q$, and by the intertwining property $\phi(q) = \phi(p \cdot g) = \phi(p) \cdot g$. By freeness, we must have $g = e$ and therefore $p = q$.

For surjectivity, let $p' \in P'$ and set $b = \pi'(p') \in B$. Choose any $p \in \pi^{-1}(b) \subset P$ and consider $\phi(p)$. This must lie in the same fiber as p' and thus $p' = \phi(p) \cdot g$ for some g, and it follows that $\phi(p \cdot g) = p'$.

To see that ϕ^{-1} is continuous, it suffices to consider local trivializations. Thus suppose $\pi^{-1}(U) \cong U \times G$ and ${\pi'}^{-1}(U) \cong U \times G$ are local trivializations of P and P' respectively, over the same set $U \subset B$ (which may be arranged by taking intersections if necessary). Then $\phi|U$ has the form

$$\phi : (b, g) \longmapsto \big(b, \phi'(b, g)\big) = \big(b, \phi'(b, e)g\big)$$

for some $\phi' : U \times G \longrightarrow G$ which satisfies $\phi'(b, gh) = \phi'(b, g)h$. Thus ϕ^{-1} has the form

$$\phi^{-1} : (b, g) \longmapsto \big(b, \phi'(b, e)^{-1}g\big)$$

which is continuous since $g \longmapsto g^{-1}$ is a continuous map on a topological group. $\square$

3.2. **Sections and trivializations.** Recall that a section of (π, P, B) over $U \subset B$ is a map $s : U \longrightarrow P$ such that $\pi \circ s = \mathrm{Id} : U \longrightarrow U$, and that a trivialization over U is a morphism $h : \pi^{-1}(U) \xrightarrow{\cong} U \times G$ intertwining multiplication on the right by G. In fact, in the category of principal bundles, these two notions are very closely related:

Proposition 3.4. *There is a bijective correspondence between sections $s \in \Gamma(U, P)$ of P over U and trivializations $h : \pi^{-1}(U) \longrightarrow U \times G$.*

Remark. Recall that the fibers of P are G-torsors — sets with a free transitive G action (hence having elements in 1–1 correspondence with $g \in G$) but without a preferred identity element. The basic idea here is that a section of P gives preferred points in the fibers which we may then identify with $e \in G$.

Proof. Given $s \in \Gamma(U, P)$, we define a trivialization as follows. For each $p \in \pi^{-1}(U)$, let $b = \pi(p) \in U$ and $g = \tau(s(b), p) \in G$. Then set

$$h : \pi^{-1}(U) \longrightarrow U \times G : p \longmapsto (b, g).$$

It is clear that this intertwines the G actions since $\tau(s(b), p \cdot g') = \tau(s(b), p)g'$, and an inverse is given by $(b, g) \longmapsto s(b) \cdot g$.

Conversely, each trivialization $h : \pi^{-1}(U) \longrightarrow U \times G$ canonically defines a section by $s(b) := h^{-1}(b, e)$, for which the construction above reproduces h. $\square$

Corollary 3.5. *A principal bundle P is globally trivial if and only if it admits global sections.*

Observe that the combination of Corollary 3.5 with the frame bundle construction Proposition 3.8 gives an alternate proof of Proposition 2.7 regarding the triviality of vector bundles.

Example 3.6. Consider the $\mathbb{Z}/2 = \{\pm 1\}$ bundle over S^1 defined by

$$P = [0, 1] \times \mathbb{Z}/2 / \{(0, +1) = (1, -1)\}.$$

Here $\mathbb{Z}/2$ is given the discrete topology. As a space it is easy to verify that this is the nontrivial double cover of S^1 and so does not admit any global sections.

As an excercise, the reader should verify that the real line bundle associated to P via the obvious multiplicative action $\mathbb{Z}/2 \longrightarrow \mathrm{GL}(\mathbb{R})$ is in fact the Möbius bundle.

3.3. **Associated bundles.** We now show how to associate a fiber bundle to a principal bundle. Let (π, P, B) be a principal G-bundle, and $\rho : G \longrightarrow \mathrm{Aut}(F)$ a left action of G on a space F. Then the product $P \times F$ has a canonical right action by

$$G \longrightarrow \mathrm{Aut}(P \times F) : (p, f) \cdot g = (p \cdot g, \rho(g^{-1})f).$$

(Recall that a left action may be turned into a right action by composing with the inverse map, and vice versa.) We will often drop the ρ from the notation and write the action on F by $f \longmapsto g^{-1} \cdot f$.

Proposition 3.7. *The quotient of $P \times F$ by G defines a fiber bundle over B with fiber space F by*

$$P \times_G F := (P \times F)/G \xrightarrow{\pi_F} B, \quad \pi_F([p, f]) := \pi(p).$$

The bundle $P \times_G F$ is called the **associated fiber bundle** *to P by $\rho : G \longrightarrow \mathrm{Aut}(F)$. If P is trivial as a G-bundle, then $P \times_G F$ is a trivial fiber bundle for any F.*

Proof. To see that π_F is well-defined, note that another representative (p', f') in the equivalence class of (p, f) is related by $p' = p \cdot g$, $f' = g^{-1} \cdot f$ for some $g \in G$, but $\pi(p \cdot g) = \pi(p)$ is therefore independent of the choice of representative.

We claim that the fibers of $P \times_G F$ are homeomorphic to F. To see this, fix a point $b \in B$ and choose a point $p_0 \in \pi^{-1}(b)$ in the fiber of P over b. We have a continuous map

$$F \longrightarrow \pi_F^{-1}(b) : f \longmapsto [p_0, f],$$

and this map has an inverse given by

$$\pi_F^{-1}(b) \longrightarrow F : [p, f] \longmapsto \tau(p_0, p) \cdot f,$$

where $\tau(p_0, p) \in G$ is the translation function (1) defined by $p_0 \cdot \tau(p_0, p) = p \in P$. Indeed, the map

$$\pi^{-1}(b) \times F \ni (p, f) \longmapsto \tau(p_0, p) \cdot f \in F$$

is invariant with respect to the G action since

$$(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f) \longmapsto \tau(p_0, p \cdot g) \cdot g^{-1} \cdot f = \tau(p_0, p)gg^{-1} \cdot f = \tau(p_0, p) \cdot f,$$

and hence descends to the quotient $\pi^{-1}(b) \times F/G = \pi_F^{-1}(b)$.

To see the local triviality, it suffices to prove the second assertion — that triviality of P implies triviality of $P \times_G F$. Thus assume that $P = B \times G$. Then

$$(P \times F)/G = (B \times G \times F)/G = \{[(b, g, f)]\} / \sim$$

This is isomorphic to $B \times F$ via $[(b, g, f)] \longmapsto (b, g \cdot f)$ with inverse map $(b, f) \longmapsto [(b, e, f)]$. □

Thus from a principal G-bundle over B, we can obtain a fiber bundle with fiber F, and whose fibers furthermore have structure group $\mathrm{Aut}(F) = G$ (the most important case of which is when F is a vector space and G acts linearly). The converse is also true.

Proposition 3.8. *Given any fiber bundle $\pi : E \longrightarrow B$ with fiber F and structure group* $\mathrm{Aut}(F)$*, there exists a principal* $\mathrm{Aut}(F)$*-bundle P such that $E = P \times_G F$.*

Remark. The principal bundle P constructed above is called the **frame bundle** of E (at least in the context of vector bundles — the terminology may be somewhat unorthodox in the setting of fiber bundles, but I think it suits!).

Proof. Set $G = \mathrm{Aut}(F)$. For $b \in B$, let P_b denote the set of 'frames': G-isomorphisms $\phi : F \longrightarrow \pi^{-1}(b)$, which is to say invertible maps $F \longrightarrow \pi^{-1}(b)$ which intertwine the action of G[1].

This has an action of G on the right, since for $g \in G$, $\phi \circ g : F \longrightarrow \pi^{-1}(b)$ is another G-isomorphism, and this action is clearly free and also transitive, since any two G-isomorphisms $\phi, \phi' : F \longrightarrow \pi^{-1}(b)$ are related by $g = \phi^{-1} \circ \phi' \in G = \mathrm{Aut}(F)$. We set

$$P = \bigcup_{b \in B} P_b$$

and $\pi_P : p \in P_b \longmapsto b$.

If $E = B \times F$ is trivial, then $\pi^{-1}(b) = \{b\} \times F$ and canonically $P_b \cong G$. Thus in this case $P = \bigcup_{b \in B} \{b\} \times G = B \times G$, and we topologize P by giving it the appropriate product topology. In the general case we do the same over local trivializations.

To see that $P \times_G F \cong E$, observe that points in $P \times_G F$ are equivalence classes $[b, \phi, f]$ where $b \in B$, $\phi : F \longrightarrow \pi^{-1}(b)$ is a G isomorphism, and $f \in F$. We consider the map

$$P \times_G F \ni [b, \phi, f] \longmapsto \phi(f) \in E.$$

This is easily seen to be well-defined since $[b, \phi \circ g, g^{-1} \cdot f] \longmapsto \phi(gg^{-1}f) = \phi(f)$, and is fiberwise isomorphic since ϕ is an isomorphism. □

Example 3.9 (Associated vector bundles)**.** If $F = V$ is a vector space, and $G \longrightarrow \mathrm{GL}(V)$ is a linear action, then $P \times_G V$ is an associated vector bundle. Conversely, each real (resp. complex) vector bundle (π, E, B) is associated to a principal $\mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$) principal bundle, or indeed to a principal $O(n)$ (resp. $U(n)$) bundle by choosing an inner product.

Example 3.10 (Associated principal bundles)**.** Consider the action of G on itself by left multiplication. This action does not preserve the group structure of G, but it does commute with right multiplication, and hence preserves the structure of G as a (right) G-torsor. Thus the associated bundle $P \times_G G$ is again a principal G-bundle, with G action $[p, g] \cdot h = [p, gh]$.

In fact $P \times_G G \cong P$. An explicit isomorphism is given by

$$P \times_G G \ni [p, g] \longmapsto p \cdot g \in P$$

This is well-defined since any other representative of $[p, g]$ has the form $(p \cdot h, h^{-1} \cdot g)$ and is mapped to $p \cdot hh^{-1}g = pg$. An inverse is given by $p \longmapsto [p, e]$.

More generally, if $\phi \in \mathrm{Aut}(G)$ is an automorphism, we can consider the left action $g \cdot h = \phi(g)h$, and the associated principal G bundle $P \times_{G,\phi} G$. In the case that $\phi(g) = \gamma g \gamma^{-1}$ is an *inner* automorphism, we again have $P \times_{G,\phi} G \cong P$. Indeed, an isomorphism is given by

$$P \times_G G \ni [p, g] \longmapsto p \cdot \gamma g \in P.$$

Any other representative $(p \cdot h, \gamma^{-1} h^{-1} \gamma g)$ maps to $p \cdot h\gamma\gamma^{-1}h^{-1}\gamma g = p \cdot \gamma g$ and an inverse is given by $p \longmapsto [p \cdot \gamma^{-1}, e]$.

Think about why this construction of an isomorphism may fail if ϕ is not inner.

[1] For the case we are most interested in, $F = \mathbb{F}^n$ and $G = \mathrm{GL}(n, \mathbb{F})$, or possibly $O(n)$ or $U(n)$ in the presence of an inner product.

Finally and more generally, to any homomorphism $\phi : G \longrightarrow H$, we consider the left G action on H by $g \cdot h = \phi(g)h$ and may form the associated principal H bundle $P \times_{G,\phi} H$.

We now show how to relate sections of an associated bundle to equivariant functions on P. If F has a left G action, we say a map $f : P \longrightarrow F$ is **equivariant** if $f(p \cdot g) = g^{-1} \cdot f(p)$. We denote the set of all such equivariant maps by $\mathrm{Map}(P, F)^G$.

Proposition 3.11. *Let (π, P, B) be a principal G-bundle, F a space with left G action, and $E = P \times_G F$ the associated bundle. There is a bijective correspondence*

$$\Gamma(U, E) \leftrightarrow \mathrm{Map}(\pi^{-1}(U), F)^G$$

Proof. Given an equivariant map $\widetilde{s} : \pi^{-1}(U) \longrightarrow F$, we define a section $s : U \longrightarrow E$ by

$$s(b) := [p, \widetilde{s}(p)], \quad \text{for some } p \in \pi^{-1}(b).$$

By the equivariance property $[p \cdot g, \widetilde{s}(p \cdot g)] = [p \cdot g, g^{-1} \cdot \widetilde{s}(p)] = [p, \widetilde{s}(p)]$, so this is well-defined.

Conversely, given a section $s : U \longrightarrow E$, we define a map $\widetilde{s} : \pi^{-1}(U) \longrightarrow F$ by $\widetilde{s}(p) = f$ where $s(\pi(p)) = [p, f]$. It follows that $\widetilde{s}(p \cdot g) = g^{-1} \cdot f$ since $s(\pi(p \cdot g)) = s(\pi(p)) = [p, f] = [p \cdot g, g^{-1} f]$. It is clear that passing from s to $\widetilde{s}$ and vice versa are inverse operations. We leave to the reader the proof that continuity of s implies continuity of $\widetilde{s}$ and vice versa. □

This has the following implication for morphisms of principal bundles:

Proposition 3.12. *Fix G and let (π, P, B) and (π', Q, B') be principal G-bundles over B and B', respectively. There is a bijective correspondence between morphisms $\phi : (\pi, P, B) \longrightarrow (\pi', Q, B')$ and sections of the associated bundle $P \times_G Q$:*

$$\mathrm{Mor}_G(P, Q) \leftrightarrow \Gamma(B, P \times_G Q).$$

Here we are regarding Q as a left G space with the action $g \cdot q := q \cdot g^{-1}$.

Proof. Recall that a morphism $\phi : (\pi, P, B) \longrightarrow (\pi', Q, B')$ is specified uniquely as a G-equivariant map $\phi : P \longrightarrow Q$. From Proposition 3.11 it therefore follows that

$$\mathrm{Mor}_G(P, Q) \equiv \mathrm{Map}(P, Q)^G \equiv \Gamma(B, P \times_G Q).$$

□

3.4. **Homotopy classification.** In this section we will discuss the homotopy classification of principal bundles. We will see that pullbacks of principal bundles by homotopic maps are isomorphic, and deduce the existence for each G of a *universal principal G-bundles* from which all other G-bundles are obtained via pullback.

The following result is of central importance in the homotopy theory of bundles.

Proposition 3.13. *If (π, P, B') is a principal G-bundle and if $f_0 \sim f_1 : B \longrightarrow B'$ are homotopic maps, then the bundles $f_0^*(P)$ and $f_1^*(P)$ over B are isomorphic.*

Proof. Considering pullback by f_t, where $f_t : B \times I \longrightarrow B'$ is the homotopy between f_0 and f_1, it suffices to show that for any G-bundle $(\pi, Q, B \times I)$, the restrictions

$$Q_0 := Q|B \times 0 \longrightarrow B \times 0 \cong B \quad \text{and} \quad Q_1 := Q|B \times 1 \longrightarrow B \times 1 \cong B$$

are isomorphic. To prove this it suffices to produce an isomorphism $Q \xrightarrow{\cong} Q_0 \times I$ of G-bundles over $B \times I$, since then restriction to $B \times 1$ gives the isomorphism $Q|B \times 1 \equiv Q_1 \xrightarrow{\cong} (Q_0 \times I)|B \times 1 \equiv Q_0$.

Thus assume given a principal G-bundle $Q \longrightarrow B\times I$, let $Q_0 = Q|B\times 0$ as above, and we will proceed to show that Q and $Q_0\times I$ are isomorphic. By Proposition 3.12, it is enough to produce a morphism $Q \longrightarrow Q_0 \times I$ over the identity on $B \times I$, since this morphism will necessarily be an isomorphism by Proposition 3.3. In turn, this is equivalent to finding a section of $Q \times_G Q_0 \times I \longrightarrow B \times I$.

Now, $Q \times_G Q_0 \times I$ has a section over $B \times 0$, since the bundles $Q|B \times 0$ and $Q_0 \times I|B \times 0 \equiv Q_0$ are isomorphic by definition.

An extension is given by the homotopy lifting property. Indeed, under the condition of paracompactness, $Q \times_G Q_0 \times I \longrightarrow B \times I$ is a fibration, and are trying to find a lift of the identity map $B \times I \longrightarrow B \times I$ to $Q \times_G Q_0 \times I$ given a map $B \times 0 \longrightarrow Q \times_G Q_0 \times I$ as in the following diagram:

$$\begin{tikzcd} B \times 0 \arrow[r] \arrow[d] & Q \times_G Q_0 \times I \arrow[d] \\ B \times I \arrow[r, "\mathrm{Id}"] \arrow[ur, dashed] & B \times I \end{tikzcd}$$

The existence of such a lift is precisely the homotopy lifting property. $\square$

For any space B, let $\mathcal{G}(B)$ denote the set of isomorphism classes of principal G-bundles over B. Observe that the assignment $B \longmapsto \mathcal{G}(B)$ is actually a contravariant (set-valued) functor. Indeed, if $f : A \longrightarrow B$ is a continuous map, then $f^* : \mathcal{G}(B) \ni P \longmapsto f^*(P) \in \mathcal{G}(A)$ is a function from $\mathcal{G}(B)$ to $\mathcal{G}(A)$. We may interpret Proposition 3.13 as saying that $\mathcal{G}$ actually descends to the homotopy category:

$$\mathcal{G} : \mathsf{hTop} \longrightarrow \{\text{principal } G\text{-bundles up to iso.}\}$$

where the morphisms in hTop are *homotopy equivalence classes* of continuous maps. The next step is to show that this functor is *representable*.

In the remainder of the section we restrict ourselves to the category CW of CW complexes. Since we will be constructing principal bundles via pullback with respect to maps defined up to homotopy, the results extend immediately to the category of spaces which are homotopy equivalent to a CW complex.

Definition 3.14. A principal G-bundle (π, EG, BG) is said to be **universal** if the total space EG is (weakly) contractible.

The name is derived from the following universal property:

Theorem 3.15. *Let (π, EG, BG) be a universal G-bundle. Then for any CW complex B, the sets $[B, BG]$ and $\mathcal{G}(B)$ are equivalent. In other words,*

$$[-, BG] \longrightarrow \mathcal{G} : [f] \longmapsto [f^*EG]$$

is an equivalence of contravariant functors $\mathsf{hCW} \longrightarrow \mathsf{Set}$. *We say that BG is a* **classifying space** *for principal G-bundles.*

Before proving Theorem 3.15, we recall the following result from the theory of CW complexes.

Lemma 3.16. *If (B, A) is a CW pair and F a space such that $\pi_k(F) = 0$ for each k such that $B \setminus A$ has cells of dimension $k+1$, then every map $f : A \longrightarrow F$ extends to a map $\widetilde{f} : B \longrightarrow F$ such that $\widetilde{f}|A \equiv f$.*

Proof. By induction on k, we may assume that f has been extended to the k-skeleton B^k of (B, A) (recall that we regard A as the -1 skeleton of B, giving the base case for the induction). For each $k+1$ cell $e^{k+1} \subset B$ with attaching map $\phi : \partial I^{k+1} \longrightarrow B^k$, the composition $f \circ \phi : \partial I^{k+1} \longrightarrow F$ is nullhomotopic by the hypothesis on F, hence can be extended to $B^k \cup_\phi e^{k+1}$. Extending f in this way for each $k+1$ cell completes the induction. □

Corollary 3.17. *Let (B, A) be a CW pair and (π, E, B) a fiber bundle with fiber F. If $\pi_k(F) = 0$ for all k such that $B \setminus A$ has cells of dimension $k+1$, then every section $s \in \Gamma(A, E)$ can be extended to a global section $\widetilde{s} :\in \Gamma(B, E)$.*

In particular, taking $A = \emptyset$, it follows that (π, E, B) admits global sections if F is k-connected where $k = \dim(B)$.

Proof. Recall that if $E = B \times F$ is trivial, then a section is equivalent to a map $B \longrightarrow F$, thus the claim follows directly from Lemma 3.16 in this case. The general case follows by refining the CW structure on B and reducing to the trivial case.

Indeed, in the general case, we proceed as above by induction on k, assuming that a section s has been extended to the k-skeleton, so $s \in \Gamma(B^k, E)$. Now a general $k+1$ cell e^{k+1} of B may not sit in a set over which E is trivial, but by subdividing $e^{k+1} \cong I^{k+1}$ into sufficiently small cubes, we may reduce to the case that $e^{k+1} \subset U_\alpha$ where $\pi^{-1}(U_\alpha) \cong U_\alpha \times F$ and the inductive step follows as before. □

Proof of Theorem 3.15. To see surjectivity, suppose $Q \longrightarrow B$ is a principal G-bundle. The associated bundle $Q \times_G EG$ has a global section over B since EG is contractible (by Corollary 3.17), which corresponds by Proposition 3.12 to a morphism $(\pi, Q, B) \longrightarrow (\pi, EG, BG)$ lying over some map $f : B \longrightarrow BG$ of the base spaces. Such a morphism is equivalent to a morphism $Q \longrightarrow f^*(EG)$ over the identiy map on B, which is therefore an isomorphism:

$$Q \cong f^*(EG), \quad f : B \longrightarrow BG.$$

To see injectivity, suppose that $f_0, f_1 : B \longrightarrow BG$ are two maps such that the pullbacks of EG are isomorphic:

$$\phi : f_0^*(EG) \xrightarrow{\cong} f_1^*(EG).$$

We claim that $f_0 \sim f_1$. Indeed, consider the principal G-bundle

$$P := f_0^*(EG) \times I \longrightarrow B \times I.$$

It is immediate that $P|B \times 0 \cong f_0^*(EG)$ and $P|B \times 1 \cong f_0^*(EG)$. The G-bundle morphism

$$\begin{tikzcd} P|B \times 0 \arrow[r, "\cong"] \arrow[dr] & f_0^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ & B \times 0 \arrow[r, "f_0"] & BG \end{tikzcd}$$

corresponds to a local section $s_0 \in \Gamma(B \times 0, P \times_G EG)$, and likewise the morphism

$$\begin{tikzcd} P|B\times 1 \arrow[r, "\phi"] \arrow[dr] & f_1^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ & B\times 1 \arrow[r, "f_1"] & BG \end{tikzcd}$$

corresponds to a local section $s_1 \in \Gamma(B \times 1, P \times_G EG)$. (Note that here we have used the given isomorphism relating $f_0^*(EG) = P|B \times 1$ and $f_1^*(EG)$.)

Putting these together, we have the section $s_0 \cup s_1 \in \Gamma(B \times 0 \cup B \times 1, P \times_G EG)$. By connectivity of EG, this extends to a global section $s \in \Gamma(B \times I, P \times_G EG)$, which therefore corresponds to a morphism $(\pi, P, B \times I) \longrightarrow (\pi, EG, BG)$ and induces a map $h : B \times I \longrightarrow BG$ which is a homotopy between $f_0 = h|B \times 0$ and $f_1 = h|B \times 1$. □

3.5. **B as a functor.** There is a general construction due to Milnor of a BG associated to any topological group G. For the applications we are interested in, we will require concrete realizations of BG, so we only sketch the proof here.

Theorem 3.18. *Given a topological group G, there exists a universal principal bundle (π, EG, BG).*

Proof sketch. For each fixed n, form the n-fold *join*

$$EG^n := G * G * \cdots * G.$$

Recall that the join of two spaces A and B is the space

$$A * B = A \times B \times I / \sim$$

where we identify all points $(a, b_1, 0) \sim (a, b_2, 0)$ and $(a_1, b, 1) \sim (a_2, b, 1)$. The resulting space can be viewed as a disjoint copy of A and B with a line segment joining each point $a \in A$ with each point $b \in B$.

It is possible to show that EG^n is $(n-1)$-connected, and it has an obvious free action by G given by right multiplication in each factor of G. Thus the limit

$$EG := \lim_{n\to\infty} EG^n$$

is a weakly contractible G-space, and $BG := EG/G$ is therefore a classifying space. □

Remark. A more proper description of this construction uses the machinery of *simplicial sets.* The space EG is the geometric realization of a natural simplicial set formed from G.

Corollary 3.19. *BG can be taken to have a CW complex structure, and such a BG is unique up to homotopy equivalence.*

Proof. Let (π, EG', BG') be any universal G bundle (say the one constructed above), and let $\phi : BG \longrightarrow BG'$ be a CW approximation. The pullback bundle $\phi^* EG'$ is seen to be weakly contractible by considering the long exact homotopy sequences of (π, EG', BG') and $(\mathrm{pr}_1, \phi^* EG', BG)$ and using the 5-lemma.

If B_1G and B_2G are two classifying spaces for G, we obtain homotopy classes of maps $f : B_1G \longrightarrow B_2G$ and $g : B_2G \longrightarrow B_1G$ classifying E_1G and E_2G

respectively. It then follows from the fact that $(f \circ g)^* E_2 G \cong E_2 G$ and $(g \circ f)^* E_1 G \cong E_1 G$ that $f \circ g \simeq 1$ and $g \circ f \simeq 1$. $\square$

In fact, $B : \mathsf{Grp} \longrightarrow \mathsf{hCW} : G \longmapsto BG$ is a functor:

Proposition 3.20. *For each homomorphism $\phi \in \mathrm{Hom}(G, H)$ there is natural homotopy class $B\phi \in [BG, BH]$ such that $B(\phi \circ \psi) = B\phi \circ B\psi$ and $B\mathrm{Id} = \mathrm{Id}$. Moreover, B preserves products in the sense that $BG \times BH$ is a $B(G \times H)$.*

Proof. The associated bundle $EG \times_{G,\phi} H$ (see Example 3.10) is a principal H-bundle over BG hence classified by a map $B\phi \in [BG, BH]$. Functoriality follows from the evident isomorphism

$$(EG \times_{G,\phi} H) \times_{H,\psi} K \cong EG \times_{G,\psi\circ\phi} K$$

and that $B\mathrm{Id} = \mathrm{Id}$ follows from the fact that $EG \times_G G \cong EG$, as proved in Example 3.10.

For the product result, simply note that $EG \times EH$ is a weakly contractible space with a $G \times H$ action with respect to which $(EG \times EH)/G \times H = BG \times BH$. $\square$

We next mention two important results concerning these induced maps which we will use in computing the cohomology of the classifying spaces for $O(n)$ and $U(n)$.

Lemma 3.21. *Let $H \subset G$ be a subgroup such that $G \longrightarrow G/H$ is a principal H bundle. Then $Bi : BH \longrightarrow BG$ can be taken to be a fiber bundle with fiber G/H.*

Remark. The condition on $H \subset G$ is satisfied in most situations of interest; in particular, if G is a Lie group then any closed subgroup has this property. In this case G/H is a so-called 'homogeneous space' (and is of course again a Lie group if H is both closed and normal).

Proof. Under the condition on G and H, the space EG is a contractible space with a free right H action, and so $(EG)/H$ is a BH, with $EH \equiv EG$. It is easy to see that $(EG)/H \cong (EG \times_G G)/H \cong EG \times_G G/H$, and we have the morphism of principal bundles

$$\begin{tikzcd}
EG \arrow[r, "\equiv"] \arrow[d] & EG \arrow[d] \\
BH = EG/H \arrow[r] & BG = EG/G
\end{tikzcd}$$

so that the induced classifying map $BH \longrightarrow BG$ may be identified with the associated bundle

$$BH = EG \times_G G/H \longrightarrow BG$$

which is a fiber bundle with fiber G/H. $\square$

Lemma 3.22. *For an inner automorphism $\phi : G \longrightarrow G$, the induced map on classifying spaces is homotopic to the identity. In other words,*

$$B\phi = \mathrm{Id} \in [BG, BG]$$

Proof. This follows from the isomorphism $EG \times_{G,\phi} G \cong EG$ constructed in Example 3.10. $\square$

4. Characteristic classes

Definition 4.1. A **characteristic class** is a functor which assigns to each vector bundle (π, E, B) a cohomology class $c(E) \in H^*(B; G)$ for some group G. Here functoriality means that for every map $f : A \longrightarrow B$, $c(f^*E) = f^*c(E) \in H^*(A; G)$.

It follows from the functoriality that $c(E) = 0$ whenever E is a trivial bundle over B, since $E = B \times \mathbb{F}^n$ is isomorphic to the pullback $f^*(\mathbb{F}^n)$ of the trivial vector bundle $\mathbb{F}^n \longrightarrow 0$ by the unique map $f : B \longrightarrow 0$, regarding 0 as a one point space. It also follows that if $E_1 \cong E_2$ as vector bundles over B, then $c(E_1) = c(E_2)$. Thus characteristic classes give necessary conditions for two bundles to be isomorphic, or for a bundle to be trivial.

One way to produce characteristic classes is to compute cohomology classes of the classifying spaces $B\mathrm{GL}(n, \mathbb{F})$, and in fact by the Yoneda lemma, all characteristic classes arise in this way.

Thus it remains to compute the cohomology of $B\mathrm{GL}(n, \mathbb{R})$ and $B\mathrm{GL}(n, \mathbb{C})$ for some groups G. Moreover, by choosing inner products, or by reduction of structure group (to be written), it suffices to consider classifying spaces for the compact groups $O(n)$ and $U(n)$.

4.1. Line Bundles.

The case $n = 1$ is rather special as we shall see. To produce a classifying space for $O(1) = \mathbb{Z}_2$, consider the quotient maps $\pi : S^n \longrightarrow \mathbb{R}P^n$. there is a free transitive right action by $\mathbb{Z}_2$ on each fiber which interchanges antipodal points and respects the inclusions

$$\begin{tikzcd} S^n \arrow[r, "\subset"] \arrow[d] & S^{n+1} \arrow[d] \\ \mathbb{R}P^n \arrow[r, "\subset"] & \mathbb{R}P^{n+1} \end{tikzcd}$$

Thus each $(\pi, S^n, \mathbb{R}P^n)$ is a principal $\mathbb{Z}_2$-bundle, as is the direct limit $S^\infty = \lim_{n\to\infty} S^n \longrightarrow \mathbb{R}P^\infty = \lim_{n\to\infty} \mathbb{R}P^n$. Since S^∞ is weakly contractible (in fact it is contractible as a CW complex), it follows that:

Proposition 4.2. *The infinite projective space $\mathbb{R}P^\infty$ is a $B\mathbb{Z}_2$, with $E\mathbb{Z}_2 = S^\infty$.*

Similarly, by considering $S^{2n+1} = \left\{ |z_0|^2 + \cdots + |z_n|^2 = 1 \right\} \subset \mathbb{C}^{n+1}$ and the quotient by the $S^1 = U(1)$ action $(z_0, \ldots, z_n) \cdot e^{i\theta} = (z_0 e^{i\theta}, \ldots, z_n e^{i\theta})$, it follows that $S^{2n+1} \longrightarrow \mathbb{C}P^n$ is a principal $U(1)$-bundle, and taking the direct limit we obtain

Proposition 4.3. *The infinite projective space $\mathbb{C}P^\infty$ is a $BU(1)$ with $EU(1) = S^\infty$.*

Problem 1. Show that with respect to the standard action $\mathbb{Z}_2 \longrightarrow \mathrm{GL}(\mathbb{R})$ the associated line bundle $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathbb{R}$ is none other than the canonical line bundle:

$$E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathbb{R} \cong \gamma^1_\infty \longrightarrow \mathbb{R}P^\infty.$$

Likewise, show that

$$EU(1) \times_{U(1)} \mathbb{C} \cong \gamma^1_\infty \longrightarrow \mathbb{C}P^\infty.$$

Definition 4.4. The generator $w_1 \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ where $H^*(\mathbb{R}P^\infty,\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$ is called the **first Steifel-Whitney class**. If (π, E, B) is a real line bundle with classifying map $f : B \longrightarrow \mathbb{R}P^\infty$, we say $w_1(E) := f^*w_1 \in H^1(B,\mathbb{Z}_2)$ is the first Steifel-Whitney class of E.

Similarly, the generator $c_1 \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ where $H^*(\mathbb{C}P^\infty,\mathbb{Z}) = \mathbb{Z}[c_1]$ is called the **first Chern class**. If (π, E, B) is a complex line bundle with classifying map $f : B \longrightarrow \mathbb{C}P^\infty$, we say $c_1(E) := f^*c_1 \in H^2(B;\mathbb{Z})$ is the first Chern class of E.

The characteristic class of a line bundle E is, by definition determined by the classifying map $f : B \longrightarrow B\mathbb{Z}_2$ or $BU(1)$. However in this instance, the converse is also true; namely, the characteristic class $w_1(E)$ or $c_1(E)$ also determines the classifying map. Indeed, we have the rather remarkable fact that $B\mathbb{Z}_2 = \mathbb{R}P^\infty$ is also a $K(\mathbb{Z}_2, 1)$ and $BU(1) = \mathbb{C}P^\infty$ is also a $K(\mathbb{Z}, 2)$. This follows in turn from the long exact homotopy sequences for the fibrations $(\pi, S^\infty, \mathbb{R}P^\infty)$ and $(\pi, S^\infty, \mathbb{C}P^\infty)$ and the fact that $\mathbb{Z}_2$ is a $K(\mathbb{Z}_2, 0)$ and S^1 is a $K(\mathbb{Z}, 1)$.

Thus for instance, $w_1(E) \in H^1(B,\mathbb{Z}_2) = [B, \mathbb{R}P^\infty]$ is represented by a unique homotopy class $f \in [B, \mathbb{R}P^\infty]$ such that $f^*w_1 = w_1(E)$ which is therefore also the classifying map for the bundle, and similarly in the complex case. We conclude that line bundles are completely classified by cohomology:

Theorem 4.5 (Classification of line bundles)**.** *The association $E \longmapsto w_1(E)$ gives a bijection between isomorphism classes of real line bundles on B and $H^1(B,\mathbb{Z}_2)$. Similarly, the association $E \longmapsto c_1(E)$ gives a bijection between isomorphism classes of complex line bundles on B and $H^2(B,\mathbb{Z})$.*

Remark. Since $H^1(B,\mathbb{Z}_2)$ and $H^2(B,\mathbb{Z})$ are also abelian groups, you might ask if there is an abelian group structure on isomorphism classes of real/complex line bundles over B such that the above bijection is a group isomorphism. In fact there is, and the group operation is given by the tensor product $(E_1, E_2) \longmapsto E_1 \otimes E_2$ of line bundles, with the trivial bundle as the identity element.

4.2. **Grassmanians.** To obtain characteristic classes for higher rank bundles, we next identify nice realizations of $BO(n)$ and $BU(n)$ as Grassmannian manifolds.

Definition 4.6. Let $V_n(\mathbb{R}^{n+k})$ denote the **Steifel manifold** of orthonormal n-tuples $(v_1, \ldots, v_n)$, $v_i \in \mathbb{R}^{n+k}$, topologized as a subspace of $\left(\mathbb{R}^{n+k}\right)^n$. There is an obvious free $O(n)$-action on $V_n(\mathbb{R}^{n+k})$, and we let $G_n(\mathbb{R}^{n+k}) = V_n(\mathbb{R}^{n+k})/O(n)$ be the **Grassmannian manifold** of n-dimensional subspaces of $\mathbb{R}^{n+k}$, with the quotient topology. The quotient is equivalent to the map sending $(v_1, \ldots, v_n)$ to the n-plane they span. Note that the fiber of the quotient is $V_n(\mathbb{R}^n)$ which is an $O(n)$-torsor.

We may similarly define the complex Steifel manifolds $V_n(\mathbb{C}^{n+k})$, and the complex Grassmanians $G_n(\mathbb{C}^{n+k}) = V_n(\mathbb{C}^{n+k})/U(n)$.

Taking the direct limit as $k \longrightarrow \infty$, we obtain the spaces $V_n(\mathbb{R}^\infty)$, $G_n(\mathbb{R}^\infty)$, and $V_n(\mathbb{C}^\infty)$, $G_n(\mathbb{C}^\infty)$.

Proposition 4.7. *The the projection $V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty)$ is a universal principal $O(n)$-bundle, and $V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$ is a universal principal $U(n)$-bundle.*

Proof. It remains to show that the quotient maps are fiber bundles, and that the total spaces are contractible. For the first claim, for any $V \in G_n(\mathbb{R}^\infty)$, define the open open neighborhood $U(V)$ to consist of all n-planes W for which the orthogonal projection $\Pi_V : W \longrightarrow V$ is an isomorphism. On fibers in $\pi^{-1}(U(V)) \subset V_n(\mathbb{R}^\infty)$, the projection $(v_1, \dots, v_n) \longmapsto (\Pi_V v_1, \dots, \Pi_V v_n)$ followed by Gram-Schmidt orthonormalization can be seen to be an $O(n)$-equivariant homeomorphism onto the fiber over V, which can be further identified with $O(n)$ by comparing with a fixed orthonormal n-frame for V. Thus $\pi^{-1}(U(V)) \cong U(V) \times O(n)$.

To see that $V_n(\mathbb{R}^\infty)$ is contractible, we apply the (injective) linear homotopy $h_t : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ where $h_t(x_1, x_2, \dots) = (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots)$ to an n-frame $(v_1, \dots, v_n)$, re-orthogonalizing for each t by Gram-Schidt. This gives a homotopy between $(v_1, \dots, v_n)$ and an n-frame all of whose vectors have vanishing x_1 coordinate. Iterating this n times gives a homotopy to an n-frame $(w_1, \dots, w_n)$ all of whose vectors have their first n coordinates vanishing, which is then homotopic by $(1-t)(w_1, \dots, w_n) + t(e_1, \dots, e_n)$ to the n-frame given by the first n standard basis vectors.

The proof in the complex case is entirely analogous. □

To compute the cohomology of $G_n(\mathbb{R}^\infty)$ and $G_n(\mathbb{C}^\infty)$, we will require the following result regarding the cohomology of certain well-behaved fiber bundles. A nice elementary proof can be found in [Hat02].

Theorem 4.8 (Leray-Hirsch)**.** *Let (π, E, B) be a fiber bundle with fiber F and let R be a PID. If $H^*(F; R)$ is a finitely generated free R-module, and if there are classes $\{c_1, \dots, c_N\} \subset H^*(E; R)$ whose restrictions $\{i^*(c_1, \dots, i^*(c_N)\}) \in H^*(F; R)$ to each fiber form a basis, then $H^*(E; R)$ is a free $H^*(B; R)$ module, with isomorphism*

$$H^*(B; R) \otimes_R H^*(F; R) \xrightarrow{\cong} H^*(E; R)$$

given by $\sum b_j i^*(c_j) \longmapsto \sum \pi^*(b_j)\, c_j$.

Remark. With respect to $\pi^* : H^*(B; R) \longrightarrow H^*(E; R)$ and the cup product, $H^*(E; R)$ always has the structure of a $H^*(B; R)$-module; the theorem gives conditions under which it is free. One can also intepret the result as saying that under the hypotheses of the theorem, E behaves cohomologically like the product $B \times F$, for which the theorem is a consequence of the Künneth and universal coefficient theorems. Recall that the isomorphism in the Leray-Hirsch theorem is not necessarily a ring isomorphism.

The fiber bundle we will consider is $Bi : BO(1)^n \longrightarrow BO(n)$. Recall that we may take $BO(1)^n = EO(n)/O(1)^n$ as the classifying space of the subgroup, which we explicitly identify as

$$V_n(\mathbb{R}^\infty)/O(1)^n = F_n(\mathbb{R}^\infty),$$

the **flag manifold** of ordered n-tuples of orthogonal lines $(L_1, \dots, L_n)$, $L_i \subset \mathbb{R}^\infty$. (Such ordered n-tuples are equivalent to so-called n-flags in $\mathbb{R}^\infty$, which are sequences $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n$ of subspaces with $\dim(V_i/V_{i-1}) = 1$.) The resulting fiber bundle

$$BO(1)^n = F_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty) = BO(n)$$

sends $(L_1, \dots, L_n)$ to the space $L_1 + \cdots + L_n$, and has fiber $F_n(\mathbb{R}^n)$, which of course is the homogeneous space $O(n)/O(1)^n$.

Theorem 4.9. *The $\mathbb{Z}_2$ cohomology of $BO(n)$ is a polynomial ring:*

$$H^*(BO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n], \quad w_i \in H^i(BO(n), \mathbb{Z}_2)$$

*The generator $w_i \in H^i(BO(n), \mathbb{Z}_2)$ is known as the i***th Steifel-Whitney class.**

Letting $p_{n,k} : O(n)\times O(k) \longrightarrow O(n+k)$ denote the inclusion of the block diagonal subgroup, the Steifel-Whitney classes satisfy

$$Bp_{n,k}^* w_j = \sum_{0\leq i\leq j} w_i\, w_{j-i} \tag{2}$$

where by convention $w_0 := 1$, and the w_i (resp. w_j) are the corresponding generators in the cohomology of $BO(n)$ (resp. $BO(k)$) or 0 if $i > n$ (resp. $j > k$).

Similarly, letting $i_n : O(n) \longrightarrow O(n+1)$, the classes satisfy

$$Bi_n^* w_j = w_j. \tag{3}$$

Proof. Note that it follows from Proposition 3.20, that the n-fold product

$$\mathbb{R}P^\infty \times \cdots \mathbb{R}P^\infty = BO(1)\times \cdots BO(1)$$

is a $BO(1)^n$, and from the Künneth theorem

$$H^*(BO(1)^n;\mathbb{Z}_2) = \mathbb{Z}_2[x_1,\ldots,x_n], \quad x_i \in H^1(BO(1)^n;\mathbb{Z}_2).$$

However, in order to apply Leray-Hirsch, we need to identify specific generators and their relation to generators of the cohomology of the fiber.

For any n and k, the $\mathbb{Z}_2$ cohomology of $F_k(\mathbb{R}^n)$ may be computed by induction using the fiber bundles

$$F_k(\mathbb{R}^n) \longrightarrow F_{k-1}(R^n), \quad (L_1,\ldots,L_k) \longmapsto (L_1,\ldots,L_{k-1}).$$

with fiber $\mathbb{R}P^{n-k}$ to obtain that $H^*(F_k(\mathbb{R}^n);\mathbb{Z}_2)$ is the quotient of the polynomial ring $\mathbb{Z}_2[x_1,\ldots,x_k]$ by the monomials $x_1^n, x_2^{n-1},\ldots,x_k^{n-k+1}$. Indeed, the classes $x_k^\alpha \in H^\alpha(F_k(\mathbb{R}^n);\mathbb{Z}_2)$ obtained by pullback from the map $F_k(\mathbb{R}^n) \longrightarrow \mathbb{R}P^{n-1}$, $(L_1,\ldots,L_k) \longmapsto L_k$ restrict to generators of the cohomology $H^*(\mathbb{R}P^{n-k},\mathbb{Z}_2)$ of the fiber, so the Leray-Hirsch theorem applies and by the inductive hypothesis

$$H^*(F_k(\mathbb{R}^n);\mathbb{Z}_2) \cong H^*(F_{k-1}(\mathbb{R}^n);\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} \mathbb{Z}_2[x_k]/x_k^{n-k+1}.$$

Letting $n \to \infty$ we again obtain the expected result that $H^*(F_n(\mathbb{R}^\infty);\mathbb{Z}_2) = \mathbb{Z}_2[x_1,\ldots,x_n]$.

Now it is clear that the generators $x_1^{\alpha_1},\ldots,x_n^{\alpha_n} \in H^*(F_n(\mathbb{R}^\infty);\mathbb{Z}_2)$ restrict to generators of the cohomology of the fiber $F_n(\mathbb{R}^n)$, so again the Leray-Hirsch theorem applies, giving

$$H^*(BO(1)^n;\mathbb{Z}_2) = H^*(BO(n);\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} H^*(F_n(\mathbb{R}^n);\mathbb{Z}_2). \tag{4}$$

in particular since $1 \in H^*(BO(1)^n;\mathbb{Z}_2)$ is a generator, there is a canonical image of $H^*(BO(n);\mathbb{Z}_2)$ in $H^*(BO(1)^n;\mathbb{Z}_2)$ as a direct summand by the map $\pi^* : H^*(BO(n);\mathbb{Z}_2) \longrightarrow H^*(BO(1)^n;\mathbb{Z}_2)$.

It is easy to see that this image lies in the symmetric polynomials in $\mathbb{Z}_2[x_1,\ldots,x_n]$ since the action of the symmetric group Σ_n by permutation of the L_i on $F_n(\mathbb{R}^\infty)$ permutes the variables x_i, but descends to act trivially on $G_n(\mathbb{R}^\infty)$. (Alternatively, the Σ_n action on $O(1)^n$ acts by inner automorphism in $O(n)$ and therefore by Lemma 3.22 induces the identity on $BO(n)$.)

To see that the image of $H^*(BO(n);\mathbb{Z}_2)$ *is exactly* the subring of symmetric polynomials, it suffices to give a counting argument using Poincaré series. The Poincaré series of $\mathbb{Z}_2[x_1]$ is $p(t) = 1 + t + t^2 + \cdots = (1-t)^{-1}$ and therefore by multiplicativity

$$p_{BO(1)^n}(t) = (1-t)^{-n}.$$

The Poincaré series of the fiber space $F_n(\mathbb{R}^n)$ is

$$p_{F_n(\mathbb{R}^n)}(t) = 1(1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = \prod_{i=1}^{n} \frac{(1-t^i)}{1-t} = (1-t)^{-n}\prod_i (1-t^i)$$

and so (4) implies that

$$p_{BO(n)}(t) = \prod_{i=1}^{n}(1-t^i)^{-1}$$

which is exactly the Poincaré series of $\mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$ where σ_i is the ith elementary symmetric polynomial of the x_j :

$$\sigma_i = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i}.$$

The multiplicativity property (2) follows from the corresponding property for elementary symmetric polynomials and the fact that $BO(1)^n \longrightarrow BO(n+k)$ factors through $Bp_{n,k} : BO(n) \times O(k) \longrightarrow BO(n+k)$.

Similarly, the naturality property (3) follows from the fact that the image of the ith elementary symmetric polynomial $\sigma_i(x_1, \ldots, x_{n+1}) \in \mathbb{Z}_2[x_1, \ldots, x_{n+1}]$ in $\mathbb{Z}_2[x_1, \ldots, x_n] = \mathbb{Z}_2[x_1, \ldots, x_{n+1}]/x_{n+1}$ is $\sigma_i(x_1, \ldots, x_n)$. □

By an essentially similar proof, replacing $\mathbb{Z}_2$ by $\mathbb{Z}$ and $\mathbb{R}$ by $\mathbb{C}$ we obtain

Theorem 4.10. *The $\mathbb{Z}$ cohomology of $BU(n)$ is a polynomial ring:*

$$H^*(BU(n); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n], \quad c_i \in H^{2i}(BU(n); \mathbb{Z})$$

*The generator $c_i \in H^{2i}(BU(n); \mathbb{Z})$ is known as the i***th Chern class.**

Letting $p_{n,k} : U(n) \times U(k) \longrightarrow U(n+k)$ denote the inclusion of the block diagonal subgroup, the Chern classes satisfy

$$Bp^*_{n,k} c_j = \sum_{0 \le i \le j} c_i \, c_{j-i} \tag{5}$$

where by convention $c_0 := 1$, and the c_i (resp. c_j) are the corresponding generators in the cohomology of $BU(n)$ (resp. $BU(k)$) or 0 if $i > n$ (resp. $j > k$).

Similarly, letting $i_n : U(n) \longrightarrow U(n+1)$, the classes satisfy

$$Bi_n^* c_j = c_j. \tag{6}$$

4.3. **Steifel-Whitney Classes.**

Translating Theorem 4.9 to the language of vector bundles, we obtain the following 'axioms' for Steifel-Whitney classes (which can be shown to characterize the classes completely).

Theorem 4.11. *For any real vector bundle (π, E, B), there exist classes $w_i(E) \in H^i(B, \mathbb{Z}_2)$, $i \in \mathbb{N}$ such that*

(a) *If $(\pi', f^*(E), A)$ is the pullback of E by $f : A \longrightarrow B$, then $w_i(f^*(E)) = f^*(w_i(E)) \in H^i(A, \mathbb{Z}_2)$.*
(b) *$w_i(E) = 0$ for $i > \operatorname{rank}(E)$.*
(c) *$w_i(E \oplus F) = \sum_{j \le i} w_j(E)\, w_{i-j}(F)$.*
(d) *$w_i(E \oplus \underline{\mathbb{R}}) = w_i(E)$.*
(e) *$w_1(\gamma^1) \ne 0 \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$.*

In light of the multiplicativity property with respect to Whitney sums, we make the following definition

Definition 4.12. For a real vector bundle (π, E, B), the **total Whitney class** is the element

$$w(E) = 1 + w_1(E) + w_2(E) + \cdots \in H^*(B, \mathbb{Z}_2).$$

This class has the followin multiplicativity property:

$$w(E \oplus F) = w(E)\, w(F).$$

4.4. **Chern Classes.**

Likewise, we have a similar axiomatic characterization of Chern classes.

Theorem 4.13. *For any complex vector bundle (π, E, B), there exist classes $c_i(E) \in H^{2i}(B, \mathbb{Z})$, $i \in \mathbb{N}$ such that*

(a) *If $(\pi', f^*(E), A)$ is the pullback of E by $f : A \longrightarrow B$, then $c_i(f^*(E)) = f^*(c_i(E)) \in H^{2i}(A, \mathbb{Z})$.*
(b) $c_i(E) = 0$ *for* $i > \operatorname{rank}_{\mathbb{C}}(E)$.
(c) $c_i(E \oplus F) = \sum_{j \leq i} c_j(E)\, c_{i-j}(F)$.
(d) $c_i(E \oplus \underline{\mathbb{C}}) = c_i(E)$.
(e) $c_1(\gamma^1) \in H^2(\mathbb{C}P^n; \mathbb{Z})$ *is a generator.*

Definition 4.14. For a complex vector bundle (π, E, B), the **total Chern class** is the element

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^*(B, \mathbb{Z}).$$

This class has the followin multiplicativity property:

$$c(E \oplus F) = c(E)\, c(F).$$

References

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