EILENBERG-ZILBER VIA ACYCLIC MODELS, AND PRODUCTS IN HOMOLOGY AND COHOMOLOGY

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1. The Eilenberg-Zilber Theorem

1.1. **Tensor products of chain complexes.** Let C_* and D_* be chain complexes. We define the tensor product complex by taking the chain space

$$C_* \otimes D_* = \bigoplus_{n\in\mathbb{Z}} (C_* \otimes D_*)_n\,, \quad (C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential defined on generators by

$$\partial_\otimes(a \otimes b) := \partial a \otimes b + (-1)^p a \otimes \partial b, \quad a \in C_p,\ b \in D_q \tag{1}$$

and extended to all of $C_* \otimes D_*$ by bilinearity. Note that the sign convention (or something similar to it) is required in order for $\partial_\otimes^2 \equiv 0$ to hold, i.e. in order for $C_* \otimes D_*$ to be a complex.

Recall that if X and Y are CW-complexes, then $X \times Y$ has a natural CW-complex structure, with cells given by the products of cells on X and cells on Y. As an exercise in cellular homology computations, you may wish to verify for yourself that

$$C_*^{\mathrm{CW}}(X) \otimes C_*^{\mathrm{CW}}(Y) \cong C_*^{\mathrm{CW}}(X \times Y).$$

This involves checking that the cellular boundary map satisfies an equation like (1) on products $a \times b$ of cellular chains.

We would like something similar for general spaces, using singular chains. Of course, the product $\Delta_p \times \Delta_q$ of simplices is not a $p+q$ simplex, though it can be subdivided into such simplices. There are two ways to do this: one way is direct, involving the combinatorics of so-called “shuffle maps,” and is somewhat tedious. The other method goes by the name of “acyclic models” and is a very slick (though nonconstructive) way of producing chain maps between $C_*(X) \otimes C_*(Y)$ and $C_*(X \times Y)$, and is the method we shall follow, following [Bre97].[1]

The theorem we shall obtain is

Theorem 1.1 (Eilenberg-Zilber)**.** *There exist chain maps*

$$\times : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y), \quad \textit{and}$$
$$\theta : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$$

which are unique up to chain homotopy, are natural in X and Y, and such that $\theta \circ \times$ and $\times \circ \theta$ are each chain homotopic to the identity.

[1] Note that Bredon uses a different sign convention for tensor products of chain maps. While his convention has some particularly nice features, notably that $\partial_\otimes = \partial \otimes 1 + 1 \otimes \partial$ can be written without an explicit sign depending on the degree of the element it is acting on, we will observe a sign convention which is consistent with [Hat02].

Corollary 1.2. *The homology of the space $X \times Y$ may be computed as the homology of the chain complex $C_*(X) \otimes C_*(Y)$:*

$$H_n(X \times Y) \cong H_n\big(C_*(X) \otimes C_*(Y)\big)$$

Note that the right hand side $H_*\big(C_*(X) \otimes C_*(Y)\big)$ is *not* generally equal to the tensor product $H_*(X) \otimes H_*(Y)$. The failure of this equality to hold is the content of the (topological) Künneth theorem, which is very similar to the universal coefficient theorem for homology, with the obstruction consisting of Tor groups $\operatorname{Tor}(H_p(X), H_q(Y))$.

1.2. **Cross product.** We will first construct the cross product

$$\times : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y).$$

It will suffice to define this on generators: given simplices $\sigma : \Delta_p \longrightarrow X$ and $\tau : \Delta_q \longrightarrow Y$, we will define the chain $\sigma \times \tau \in C_{p+q}(X \times Y)$. Observe that when either p or q is 0 there is an obvious way to do this. Indeed, if σ is a 0-simplex, its image is just some point $x \in X$, and for each $x \in X$ there is a unique such singular 0-simplex, which we will (abusively) denote as x:

$$x : \Delta_0 \longrightarrow x \in X$$

If $\tau : \Delta_q \longrightarrow Y$ is any q-simplex on Y, then

$$x \times \tau : \Delta_q \cong \Delta_0 \times \Delta_q \longrightarrow x \times \tau(\Delta_q) \subset X \times Y \tag{2}$$

is a q-simplex on $X \times Y$. Similarly, $\sigma \times y : \Delta_p \longrightarrow X \times Y$ is defined for any p-simplex σ on X and 0-simplex $y \in Y$.

Proposition 1.3. *For any X and Y there exists a chain map $\times : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ (which we will denote by $a \times b := \times(a \otimes b)$) such that*

(i) *$\times$ coincides with the natural map* (2) *when one factor is a 0-chain.*

(ii) *With respect to the differentials, $\times$ satisfies*

$$\partial(a \times b) = \partial a \times b + (-1)^{|a|} a \times \partial b. \tag{3}$$

(iii) *$\times$ is natural in X and Y; in other words if $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are continuous maps, then*

$$(f \times g)_\#(a \times b) = (f_\# a) \times (g_\# b). \tag{4}$$

Remark. The trick here, called the "method of acyclic models," is that it suffices to consider a very special case, namely when $X = \Delta_p$ and $Y = \Delta_q$ are themselves simplices, and the chains on X and Y are the identity maps $i_p : \Delta_p \longrightarrow \Delta_p$ and $i_q : \Delta_q \longrightarrow \Delta_q$ thought of as singular simplices (these are the "models," the "acyclic" part refers to the fact that $\Delta_p \times \Delta_q$ has trivial homology, being contractible.)

In the induction step, to define $i_p \times i_q$ we *formally* compute its boundary, using property (ii). This gives a chain which we compute to be a cycle. "Of course it is a cycle," you say, "it is a boundary!" But this not correct since the thing it is supposed to be a boundary *of*, namely $i_p \times i_q$ has not yet been defined! However, since $\Delta_p \times \Delta_q$ is acyclic (has trivial homology groups), any cycle must be a boundary of some chain, and we then *define* $i_p \times i_q$ to be this chain. The definition of $\sigma \times \tau$ for general p- and q-chains on spaces X and Y is then forced by naturality.

Proof. We define $\times : C_p(X) \otimes C_q(Y) \longrightarrow C_{p+q}(X \times Y)$ by induction on $n = p+q$. The base case $n = 1$ is determined by property (i) above.

Thus suppose $\times$ has been defined on chains of degree p and q for arbitrary spaces, for all $p + q \leq n - 1$. Suppose now $p + q = n$, and let

$$i_p : \Delta_p \longrightarrow \Delta_p, \quad i_q : \Delta_q \longrightarrow \Delta_q$$

be the identity maps, but viewed as a singular p- and q-simplices on the spaces Δ_p and Δ_q, respectively. These are the "models," and we will first define $i_p \times i_q \in C_*(\Delta_p \times \Delta_q)$.

Were $i_p \times i_q$ to be defined, its boundary would have to be

$$\text{“}\partial(i_p \times i_q)\text{”} = \partial i_p \times i_q + (-1)^p i_p \times \partial i_q \tag{5}$$

by property (ii). The left-hand side is not yet defined, but the right-hand side is a well-defined chain in $C_{n-1}(\Delta_p \times \Delta_q)$ by the induction hypothesis. We observe that the right-hand side of (5) is a cycle:

$$\partial(\text{RHS}) = \partial^2 i_p \times i_q + (-1)^{p-1} \partial i_p \times \partial i_q + (-1)^p \partial i_p \times \partial i_q + i_p \times \partial^2 i_q = 0$$

and since $H_{n-1}(\Delta_p \times \Delta_q) = 0$, this cycle is the boundary of some chain:

$$\text{RHS} = \partial\alpha, \quad \alpha \in C_n(\Delta_p \times \Delta_q)$$

We take $i_p \times i_q$ to be this chain:

$$i_p \times i_q := \alpha \in C_n(\Delta_p \times \Delta_q).$$

Now suppose we have singular simplices $\sigma : \Delta_p \longrightarrow X$ and $\tau : \Delta_q \longrightarrow Y$. Viewed as continuous maps of spaces, these induce chain maps

$$\sigma_\# : C_*(\Delta_p) \longrightarrow C_*(X), \quad \tau_\# : C_*(\Delta_q) \longrightarrow C_*(Y)$$

and we observe that, *as chains*, σ and τ can be written as pushforwards of the models:

$$\sigma = \sigma_\#(i_p) : \Delta_p \longrightarrow X, \quad \tau = \tau_\#(i_q) : \Delta_q \longrightarrow Y.$$

(This is so tautologous as to be somewhat confusing! Make sure you understand what is going on here.) Property (iii) forces us to define

$$\sigma \times \tau = \sigma_\#(i_p) \times \tau_\#(i_q) = (\sigma \times \tau)_\#(i_p \times i_q).$$

We verify that this satisfies property (ii):

$$\begin{aligned}
\partial(\sigma \times \tau) &= \partial(\sigma \times \tau)_\#(i_p \times i_q) \\
&= (\sigma \times \tau)_\#\big(\partial(i_p \times i_q)\big) \\
&= (\sigma \times \tau)_\# \left(\partial i_p \times i_q\right) + (-1)^p (\sigma \times \tau)_\# \left(i_p \times \partial i_q\right) \\
&= \sigma_\#(\partial i_p) \times \tau_\#(i_q) + (-1)^p \sigma_\#(i_p) \times \tau_\#(\partial i_q) \\
&= \partial\sigma_\#(i_p) \times \tau_\#(i_q) + (-1)^p \sigma_\#(i_p) \times \partial\tau_\#(i_q) \\
&= \partial\sigma \times \tau + (-1)^p \sigma \times \partial\tau.
\end{aligned}$$

Extending $\times$ bilinearly to chains completes the induction. □

1.3. **The dual product.** Next we define $\theta : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$. Once again, there is an obvious definition on 0-chains namely, if

$$(x, y) : \Delta_0 \longrightarrow (x, y) \in X \times Y$$

is a 0-simplex, which we identify with its image in $X \times Y$, we should take

$$\theta(x, y) = x \otimes y \in C_0(X) \otimes C_0(Y) \tag{6}$$

We again use acyclic models, defining θ first on the model simplices $d_n : \Delta_n \longrightarrow \Delta_n \times \Delta_n$ given by the diagonal inclusion $d_n(v) = (v, v)$. We shall require the following lemma, which gives the acyclicity of the chain complexes $C_*(\Delta_n) \otimes C_*(\Delta_n)$.

Lemma 1.4. *If X and Y are contractible spaces, then*

$$H_n\big(C_*(X) \otimes C_*(Y)\big) = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0. \end{cases}$$

Proof. First we recall a construction giving a chain contraction of $C_*(X)$. Let $F : X \times I \longrightarrow X$ be a homotopy between the identity $F(\cdot, 0) = \mathrm{Id}$ and the contraction to a point $F(\cdot, 1) = x_0 \in X$. We will define a chain homotopy $D : C_*(X) \longrightarrow C_{*+1}(X)$ such that

$$\mathrm{Id} - \epsilon = D\partial + \partial D$$

where ϵ is the chain map $C_*(X) \longrightarrow C_*(X)$ which is the zero map in all nonzero degrees, and the augmentation map $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i \in \mathbb{Z}$ in degree 0.

Recall that one canonical construction of the n simplex Δ_n is as the set of points

$$\Delta_n = \Big\{ \sum_{i=0}^{n} t_i e_i \;\Big|\; \sum_i t_i = 1 \Big\} \subset \mathbb{R}^{n+1}$$

where $\{e_0, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^{n+1}$. With this description we can regard $(t_0, \ldots, t_n)$ as "coordinates" Δ_n, which are overdetermined since $\sum_i t_i = 1$. In particular, the faces of Δ_n are given by $\{t_i = 0\} : i = 0, \ldots, n$ and the vertices are given by $\{t_i = 1\}$.

Given a singular n-simplex $\sigma : \Delta_n \longrightarrow X$, we define $D(\sigma)$ to be the $(n+1)$-simplex

$$D(\sigma)(t_0, \ldots, t_{n+1}) = F\big(\sigma(t_1, \ldots, t_n), t_0\big) : \Delta_{n+1} \longrightarrow X$$

Observe that the face $\Delta_n \cong \{t_0 = 0\} \subset \Delta_{n+1}$ is mapped onto $\sigma(\Delta_n)$ and the vertex $\{t_0 = 1\}$ is mapped onto the contraction point x_0.

If σ has degree ≥ 1, one can check that

$$\partial D(\sigma) = \sigma - D(\partial\sigma)$$

and if σ has degree 0 then

$$\partial D(\sigma) = \sigma - x_0$$

where we identify x_0 and the 0-simplex with image $x_0 \in X$. Thus $\partial D + D\partial = \mathrm{Id} - \epsilon$ where ϵ is 0 in nonzero degrees and $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i x_0$ can be identified with the augmentation map in degree 0.

Now since X and Y are both contractible, we have such chain homotopies for each complex $C_*(X)$ and $C_*(Y)$. It suffices to combine them somehow into a chain homotopy of $C_*(X) \otimes C_*(Y)$ from $\mathrm{Id} \otimes \mathrm{Id}$ to $\epsilon \otimes \epsilon$, for then $H_*\big(C_*(X) \otimes C_*(Y)\big)$ will be equal to the homology of the image under $\epsilon \otimes \epsilon$, which is a trivial complex with only a copy of $\mathbb{Z}$ in degree 0.

Define $Q : (C_*(X) \otimes C_*(Y))_* \longrightarrow (C_*(X) \otimes C_*(Y))_{*+1}$ by

$$Q(a \otimes b) = (D \otimes \epsilon)(a \otimes b) + (-1)^{|a|}(1 \otimes D)(a \otimes b) = D(a) \otimes \epsilon(b) + (-1)^{|a|} a \otimes D(b).$$

We compute $\partial_\otimes Q + Q\partial_\otimes$, acting on an element $a \otimes b$ (which we will omit)

$$\begin{aligned}
\partial_\otimes Q + Q\partial_\otimes &= \partial D \otimes \epsilon + (-1)^{|a|+1} D \otimes \partial\epsilon + (-1)^{|a|}\partial \otimes D + (-1)^{2|a|} 1 \otimes \partial D \\
&\quad + D\partial \otimes \epsilon + (-1)^{|a|} D \otimes \epsilon\partial + (-1)^{|a|-1}\partial \otimes D + (-1)^{2|a|} 1 \otimes D\partial \\
&= (\partial D + D\partial) \otimes \epsilon + 1 \otimes (\partial D + D\partial) \\
&= (1 - \epsilon) \otimes \epsilon + 1 \otimes (1 - \epsilon) \\
&= 1 \otimes 1 - \epsilon \otimes \epsilon.
\end{aligned}$$

Thus Q is a chain homotopy between $\mathrm{Id} = 1 \otimes 1$ and $\epsilon \otimes \epsilon$. □

Proposition 1.5. *For any X and Y, there exists a chain map $\theta : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$ such that*

(i) *θ is given by (6) on 0-chains.*
(ii) *$\partial_\otimes \circ \theta = \theta \circ \partial$.*
(iii) *If $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are continuous maps, then*

$$\theta \circ (f \times g)_\# = (f_\# \otimes g_\#) \circ \theta.$$

Proof. The proof is by acyclic models. By induction, suppose that such $\theta : C_k(X \times Y) \longrightarrow \big(C_*(X) \otimes C_*(Y)\big)_k$ has been defined for chains of degree $k \le n-1$. (Property (i) furnishes the base case $k = 0$.)

Consider the product space $\Delta_n \times \Delta_n$ and let

$$d_n : \Delta_n \longrightarrow \Delta_n \times \Delta_n$$

denote the diagonal inclusion, viewed as a singular n-simplex. In order to define $\theta(d_n)$ we compute its formal boundary

$$\text{“}\partial\theta(d_n)\text{”} = \theta(\partial d_n) \in \big(C_*(\Delta_n) \otimes C_*(\Delta_n)\big)_{n-1}. \tag{7}$$

The right-hand side is a well-defined chain by the induction hypothesis, which is a cycle since

$$\partial\big(\theta(\partial d_n)\big) = \theta(\partial^2 d_n) \equiv 0$$

By Lemma 1.4 the chain complex $C_*(\Delta_n) \otimes C_*(\Delta_n)$ has trivial homology groups (except in degree 0, but in the case $n = 1$, it can be seen that the right-hand side of (7) maps to 0 by the augmentation map, hence its homology class is 0) so the right-hand side of (7) is a boundary

$$\theta(\partial d_n) = \partial_\otimes \beta, \quad \beta \in \big(C_*(\Delta_n) \otimes C_*(\Delta_n)\big)_n$$

and we define $\theta(d_n) := \beta$.

For a general product space $X \times Y$ with singular n-simplex $\sigma : \Delta_n \longrightarrow X \times Y$, composition with the projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ gives maps

$$\pi_X\sigma : \Delta_n \longrightarrow X, \quad \pi_Y\sigma : \Delta_n \longrightarrow Y$$

and we consider the chain map

$$(\pi_X\sigma \times \pi_Y\sigma)_\# : C_*(\Delta_n \times \Delta_n) \longrightarrow C_*(X \times Y)$$

induced by the product $\pi_X\sigma \times \pi_Y\sigma : \Delta_n \times \Delta_n \longrightarrow X \times Y$. Observe that, as an n-chain, σ is given by the composition

$$\sigma = (\pi_X\sigma \times \pi_Y\sigma)_\# (d_n) : \Delta_n \longrightarrow X \times Y$$

Thus the naturality property (iii) forces the definition

$$\theta(\sigma) = \theta (\pi_X\sigma \times \pi_Y\sigma)_\# (d_n) := (\pi_X\sigma)_\# \otimes (\pi_Y\sigma)_\# \theta(d_n).$$

We verify that this satisfies property (ii):

$$\begin{aligned}
\partial_\otimes\theta(\sigma) &= \partial_\otimes\big((\pi_X\sigma)_\# \otimes (\pi_Y\sigma)_\# \theta(d_n)\big) \\
&= (\pi_X\sigma)_\# \otimes (\pi_Y\sigma)_\# \partial_\otimes\theta(d_n) \\
&= (\pi_X\sigma)_\# \otimes (\pi_Y\sigma)_\# \theta(\partial d_n) \\
&= \theta (\pi_X\sigma \times \pi_Y\sigma)_\# (\partial d_n) \\
&= \theta\partial (\pi_X\sigma \times \pi_Y\sigma)_\# (d_n) \\
&= \theta\partial\sigma.
\end{aligned}$$

Extending θ linearly to chains completes the induction. □

1.4. **Chain homotopies.** Our constructions of $\times$ and θ above involved noncanonical choices (of chains whose boundary was a given chain, for instance) so we must show that, up to chain homotopy, the particular choice made is irrelevant. We will also show that $\times$ and θ are inverses up to chain homotopy. Both of these facts follow from the next proposition.

Proposition 1.6. *Any two natural chain maps from $C_*(X \times Y)$ to itself, or from $C_*(X) \otimes C_*(Y)$ to itself, or from one of these to the other, which are the canonical ones in degree 0, are chain homotopic.*

Proof. The proof (once again via acyclic models) in all four cases is essentially the same: one defines the chain homotopy map D by induction, constructing it first on the models $i_p \otimes i_q \in C_p(\Delta_p) \otimes C_q(\Delta_q)$ or $d_n \in C_n(\Delta_n \times \Delta_n)$ and then on general chains by naturality. To illustrate how it goes, we will present one of the cases in detail and leave the others to the reader. See [Bre97] for another of the cases.

Suppose ϕ and ψ are two chain maps

$$\phi, \psi : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X) \otimes C_*(Y)$$

which are natural in X and Y and equal to the identity on 0 chains. Define $D : (C_*(X) \otimes C_*(Y))_* \longrightarrow (C_*(X) \otimes C_*(Y))_{*+1}$ to be the zero map on 0 chains, and by induction assume D has been defined on chains of degree $\leq n-1$ and naturally in X and Y, so that

$$\partial D = \phi - \psi - D\partial. \tag{8}$$

Let $p + q = n$. To define $D(i_p \otimes i_q) \in (C_*(\Delta_p) \otimes C_*(\Delta_q))_{n+1}$ we compute

$$\begin{aligned}
\partial (\phi - \psi - D\partial) (i_p \otimes i_q) &= (\phi\partial - \psi\partial - (\partial D)\partial) (i_p \otimes i_q) \\
&= (\phi\partial - \psi\partial - (\phi - \psi - D\partial)\partial) (i_p \otimes i_q) \\
&= \big(\phi\partial - \psi\partial - \phi\partial + \psi\partial + D\partial^2\big) (i_p \otimes i_q) \\
&\equiv 0 \in (C_*(\Delta_p) \otimes C_*(\Delta_q))_n
\end{aligned}$$

so $(\phi - \psi - D\partial)(i_p \otimes i_q)$ is a cycle, hence a boundary $\partial\beta$ for some $\beta \in (C_*(\Delta_p) \otimes C_*(\Delta_q))_{n+1}$ and we set $D(i_p \otimes i_q) = \beta$. Thus D satisfies (8).

For $\tau \otimes \sigma \in C_p(X) \otimes C_q(Y)$ we then define

$$D(\tau \otimes \sigma) = (\tau_\# \otimes \sigma_\#)\big(D(i_p \otimes i_q)\big).$$

whence D is natural in X and Y. Since ϕ, ψ and ∂ are also natural, (8) holds and this completes the inductive step. □

Theorem 1.1 is now a direct consequence of Propositions 1.3, 1.5 and 1.6.

We shall require one other consequence of this chain homotopy result, which determines the effect of switching the factors in the external product. Let $T : X \times Y \longrightarrow Y \times X$ be the obvious transposition map. Since $T^2 = \mathrm{Id}$ this map induces an isomorphism

$$T_\# : C_*(X \times Y) \longrightarrow C_*(Y \times X)$$

and similarly on homology. On the other side, we consider the transition map

$$\begin{aligned} \tau :&C_*(X) \otimes C_*(Y) \longrightarrow C_*(Y) \otimes C_*(X) \\ &\alpha \otimes \beta \longmapsto (-1)^{|\alpha||\beta|} \beta \otimes \alpha \end{aligned}$$

Note that the sign is required in order for τ to be a chain map (so that $\partial_\otimes \tau = \tau \partial_\otimes$), which is readily verified. This also satisfies $\tau^2 = \mathrm{Id}$ and thus is also an isomorphism.

Consider the diagram

$$\begin{tikzcd} C_*(X) \otimes C_*(Y) \arrow[r, "\times"] \arrow[d, "\tau"] & C_*(X \times Y) \arrow[d, "T_\#"] \\ C_*(Y) \otimes C_*(X) \arrow[r, "\times"] & C_*(X \times Y) \end{tikzcd} \tag{9}$$

The diagram is *not* commutative in general, but from Proposition 1.6 we conclude

Corollary 1.7. *The maps $\times$ and $T_\#^{-1} \circ \times \circ \tau$ are chain homotopic. Similarly, in the noncommutative diagram*

$$\begin{tikzcd} C_*(X) \otimes C_*(Y) \arrow[d, "\tau"] & C_*(X \times Y) \arrow[l, "\theta"'] \arrow[d, "T_\#"] \\ C_*(Y) \otimes C_*(X) & C_*(X \times Y) \arrow[l, "\theta"'] \end{tikzcd} \tag{10}$$

the maps θ and $\tau^{-1}\theta T_\#$ are chain homotopic.

Proof. The maps are natural in X and Y and are the obvious ones,

$$x_0 \otimes y_0 \longrightarrow x_0 \times y_0$$

and

$$x_0 \times y_0 \longrightarrow x_0 \otimes y_0$$

in degeree 0. □

2. Cross product in homology and the Künneth theorem

Observe that since $\times : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ is unique up to homotopy and satisfies

$$\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^{|\sigma|}\sigma \times \partial\tau$$

it descends to a well defined *cross product*

$$\times : H_p(X) \otimes H_q(Y) \longrightarrow H_{p+q}(X \times Y). \tag{11}$$

Indeed, choosing representative cycles σ and τ for the homology classes $[\sigma]$ and $[\tau]$, it follows that

$$[\sigma] \times [\tau] := [\sigma \times \tau]$$

is independent of choices; for instance if σ' is another choice with $\sigma - \sigma' = \partial\gamma$ we have

$$\sigma \times \tau - \sigma' \times \tau = (\partial\gamma) \times \tau = \partial(\gamma \times \tau)$$

since $\partial\tau = 0$.

Note that if we identify $H_*(X \times Y)$ and $H_*(Y \times X)$ with respect to the isomorphism T_* in section 1.4 above, it follows from Corollary 1.7 that the cross product is *graded commutative*

$$a \times b = (-1)^{|a||b|} b \times a \in H^*(X \times Y).$$

It is also natural with respect to maps; if $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are continuous, then

$$(f_*a) \times (g_*b) = (f \times g)_*(a \times b) \in H^*(X' \times Y')$$

where $f \times g : X \times Y \longrightarrow X' \times Y'$ is the product map.

Also, if $A \subset X$ we note that the cross product carries $C_*(A) \otimes C_*(Y)$ into $C_*(A \times Y)$ and therefore $C_*(X, A) \otimes C_*(Y)$ into $C_*(X \times Y, A \times Y)$. This induces relative products

$$\begin{aligned} &\times : H_p(X, A) \otimes H_q(Y) \longrightarrow H_{p+q}(X \times Y, A \times Y) \\ &\times : H_p(X, A) \otimes H_q(Y, B) \longrightarrow H_{p+q}(X \times Y, A \times Y \cup X \times B) \end{aligned}$$

Summing over all p and q such that $p + q = n$ we can consider the total map

$$\times : \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y)$$

and ask whether it is an isomorphism. The answer, which is that it is always injective but not necessarily surjective, is quantified by the Künneth theorem, which we treat next. We will give a completely algebraic version first.

Theorem 2.1 (Algebraic Künneth theorem)**.** *Let C_* and C'_* be free chain complexes. Then for each n there are short exact sequences*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}\big(H_p(C_*), H_q(C'_*)\big) \longrightarrow 0 \tag{12}$$

which are natural in C_* and C'_* and which split, though not naturally.

Proof. First consider the case that C'_* has trivial differential, so that $H_*(C'_*) = C'_*$ and $\partial_\otimes = \partial \otimes 1$ on $C_* \otimes C'_*$. The homology groups $H_n(C_* \otimes C'_*)$ then consist of

$$H_n(C_* \otimes C'_*) = \frac{\operatorname{Ker}\{\partial \otimes 1 : (C_* \otimes C'_*)_n \longrightarrow (C_* \otimes C'_*)_{n-1}\}}{\operatorname{Im}\{\partial \otimes 1 : (C_* \otimes C'_*)_{n+1} \longrightarrow (C_* \otimes C'_*)_n\}}$$

and since $\partial \otimes 1$ preserves the direct sum decomposition

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

it follows that

$$H_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} H_p(C_* \otimes C'_q) = \bigoplus_{p+q=n} H_p(C_*) \otimes C'_q \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*)$$

where we used the Universal Coefficient Theorem, the fact that C'_q is free and the identification $C'_q = H_q(C_*)$. Since Tor vanishes on free groups, this establishes the theorem in this case.

Now consider the exact sequence of chain complexes

$$0 \longrightarrow Z'_* \longrightarrow C'_* \longrightarrow B'_{*-1} \longrightarrow 0$$

We apply the functor $C_* \otimes -$ (which is exact since all groups involved are free) to obtain

$$0 \longrightarrow C_* \otimes Z'_* \longrightarrow C_* \otimes C'_* \longrightarrow C_* \otimes B'_{*-1} \longrightarrow 0 \tag{13}$$

which generates the long exact sequence

$$\cdots \to H_{n+1}(C_* \otimes B'_{*-1}) \stackrel{1\otimes i}{\to} H_n(C_* \otimes Z'_*) \to H_n(C_* \otimes C'_*) \to H_n(C_* \otimes B'_{*-1}) \stackrel{1\otimes i}{\to} H_{n-1}(C_* \otimes Z'_*) \to \cdots$$

(The notation for the connecting homomorphism will be explained in a moment.) Since Z'_* and B'_* are trivial free complexes, we have

$$H_n(C_* \otimes Z'_*) = \bigoplus_{p+q=n} H_p(C_*) \otimes Z'_q$$

and

$$H_n(C_* \otimes B'_{*-1}) = \bigoplus_{p+q=n} H_p(C_*) \otimes B'_{q-1} = \bigoplus_{p+q=n-1} H_p(C_*) \otimes B'_q$$

by the first part of the proof.

We claim that the connecting homomorphism $H_n(C_* \otimes B'_{*-1}) \longrightarrow H_{n-1}(C_* \otimes Z'_*)$ in the long exact sequence above is just the map

$$1 \otimes i : \bigoplus_{p+q=n-1} H_p(C_*) \otimes B'_q \longrightarrow \bigoplus_{p+q=n-1} H_p(C_*) \otimes Z'_q$$

induced by the inclusion $B'_* \subset Z'_*$. Indeed, a cycle of degree n in the rightmost group of (13) is a sum of elements of the form $\alpha \otimes \beta$ with $\partial\alpha = 0$ (since the differential there is just $\partial \otimes 1$). The connecting homomorphism lifts this leftward to $\alpha \otimes \gamma$ such that $\partial\gamma = \beta$, then applies $\partial_\otimes$ which results in $\alpha \otimes \beta$ again (since $\partial\alpha = 0$) which is in $C_* \otimes Z'_*$, proving the claim.

For each n there is therefore the short exact sequence

$$0 \longrightarrow \operatorname{Coker}(1 \otimes i)_{n+1} \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow \operatorname{Ker}(1 \otimes i)_n \longrightarrow 0$$

extracted from (13). We will obtain the Künneth sequence once we identify these groups.

Fixing degrees for the moment, consider the sequence

$$0 \longrightarrow B'_q \longrightarrow Z'_q \longrightarrow H_q(C'_*) \longrightarrow 0.$$

We apply $H_p(C_*) \otimes -$ which is *not* exact in general; from the theory of Tor groups we get the sequence

$$0 \longrightarrow \mathrm{Tor}\big(H_p(C_*), H_q(C'_*)\big) \longrightarrow H_q(C_*)\otimes B'_q \xrightarrow{1\otimes i} H_q(C_*)\otimes Z'_q \longrightarrow H_p(C_*)\otimes H_q(C'_*) \longrightarrow 0$$

which identifies the kernel and cokernel of $1 \otimes i$. Summing over $p+q = n$ gives (12).

The existence of a splitting for (12) follows from the existence of splittings of (13) and the analogous sequence for C_* and will be left as an exercise to the reader. $\square$

Combining the algebraic Künneth theorem with the Eilenberg-Zilber Theorem 1.1 we obtain

Corollary 2.2 (Geometric Künneth theorem)**.** *For spaces X and Y, and for each n, there are short exact sequences*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X)\otimes H_q(Y) \xrightarrow{\times} H_n(X\times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}\big(H_p(X), H_q(Y)\big) \longrightarrow 0$$

which are natural in X and Y and which split, though not naturally.

We get a similar theorem relating groups $H_*(X, A)$, $H_*(Y)$ and $H_*(X\times Y, A\times Y)$ from the relative version of $\times$ and the five lemma, but beware that there is *not* a general version of the Künneth theorem in which both spaces are replaced by pairs. Such a theorem holds in certain cases, such as when A and B are both open sets (see [Bre97]), or in the case that A and B are both basepoints, which gives a Künneth theorem relating reduced homology groups $\widetilde{H}_*(X)$, $\widetilde{H}_*(Y)$ and $\widetilde{H}_*(X \wedge Y)$. (To prove this version one can just use augmented singular chain complexes for X and Y.)

3. Cross product in cohomology

Let R be a ring and consider the singular cochains $C^*(X; R)$, $C^*(Y; R)$. The map $\theta : C_p(X \times Y) \longrightarrow C_q(X) \otimes C_*(Y)$ defines a dual map

$$\theta^* : C^p(X; R) \otimes C^q(Y; R) \longrightarrow C^{p+q}(X \times Y; R \otimes R),$$

where if $f \in C^p(X; R)$ and $g \in C^q(Y; R)$, the element $\theta^* f \otimes g$ acts on a chain $\alpha \in C_{p+q}(X \times Y)$ by $(\theta^* f \otimes g)(\alpha) = f \otimes g(\theta\alpha) \in R \otimes R$. We can further compose this with the ring multiplication $\mu : R \otimes R \longrightarrow R$ to get a bilinear map

$$\times := \mu \circ \theta^* : C^p(X; R) \otimes C^q(Y; R) \longrightarrow C^{p+q}(X \times Y; R)$$

which we call the *cross product* on cochains. It is straightforward to verify that the differential is a graded derivation with respect to this product:

$$\delta(f \times g) = \delta f \times g + (-1)^{|f|} f \times \delta g$$

It thus defines a well-defined *cross product* on cohomology

$$\times : H^p(X; R) \otimes H^q(Y; R) \longrightarrow H^{p+q}(X \times Y; R)$$

which is unique since various choices of θ are chain homotopic. If R is commutative, it follows from Corollary 1.7 that, as for the cross product in homology, the cohomology cross product is graded commutative:

$$[f] \times [g] = (-1)^{|f||g|}[g] \times [f].$$

One relationship between the cross products in cohomology and homology is the following. Recall that there is a well-defined map $H^*(X; R) \longrightarrow \mathrm{Hom}(H_*(X); R)$

which evaluates representative cochains on representative chains. It is easy to check that, if $[f] \in H^p(X;R)$, $[g] \in H^q(Y;R)$, $[\alpha] \in H_p(X)$ and $[\beta] \in H_q(Y)$, then we have

$$([f] \times [g])([\alpha] \times [\beta]) = f(\alpha)g(\beta) \in R.$$

Similar to the situation in homology, the cross product has well-defined relative and reduced versions:

$$\begin{aligned}
\times &: H^p(X,A;R) \otimes H^q(Y;R) \longrightarrow H^{p+q}(X \times Y, A \times Y; R) \\
\times &: H^p(X,A;R) \otimes H^q(Y,B;R) \longrightarrow H^{p+q}(X \times Y, A \times Y \cup X \times B; R) \\
\times &: \widetilde{H}^p(X;R) \otimes \widetilde{H}^q(Y;R) \longrightarrow \widetilde{H}^{p+q}(X \wedge Y; R)
\end{aligned}$$

There is a distinguished element $1 \in H^0(X;R)$ for any space X and ring (with identity) R, given by the cohomology class of the augmentation cocycle, i.e. the cocycle which takes the value $1 \in R$ on all 0-simplices in X.

Lemma 3.1. *For any $a \in H^p(Y;R)$, the following identity holds:*

$$1 \times a = p_Y^*(a) \in H^p(X \times Y; R)$$

where $p_Y : X \times Y \longrightarrow Y$ is the projection.

Proof sketch. Since $1 \in H^0(X;R)$ is the image of $1 \in H^0(x_0;R)$ under the pullback by the projection map $X \longrightarrow x_0$, where $x_0 \in X$ is any point, it suffices to consider the case that $X = x_0$ is a single point.

Let $a \in C^*(Y;R) = \mathrm{Hom}\big(C_*(X);R\big)$ be a representative cocycle. Then $1 \times a$ is given by composing a with the sequence

$$C_*(x_0 \times Y) \xrightarrow{\theta} C_*(x_0) \otimes C_*(Y) \xrightarrow{\epsilon\otimes 1} \mathbb{Z} \otimes C_*(Y) \cong C_*(Y) \tag{14}$$

whereas $p_Y^* a$ is given by composing a with

$$C_*(x_0 \times Y) \xrightarrow{(p_Y)_*} C_*(Y). \tag{15}$$

Easy acyclic model arguments show that (14) and (15) are chain homotopic. □

3.1. **Cup product.** Having defined the cross product in cohomology, we may construct the cup product as follows. Let

$$d : X \longrightarrow X \times X$$

be the inclusion of the diagonal. Then for $a, b \in H^*(X;R)$ the product

$$a \smile b := d^*(a \times b) \in H^*(X;R)$$

defines a natural bilinear map

$$\smile : H^p(X;R) \otimes H^q(X;R) \longrightarrow H^{p+q}(X;R)$$

which is graded commutative if R is commutative:

$$a \smile b = (-1)^{|a||b|} b \smile a.$$

Here naturality with respect to a map $f : X \longrightarrow X'$ means that

$$f_*(a \smile b) = (f_* a) \smile (f_* b).$$

Assuming henceforth that R is a commutative ring with identity, the cup product makes $H^*(X;R)$ into a graded commutative ring with identity $1 \in H^0(X;R)$ as defined above. To see that 1 is the identity observe that

$$1 \smile a = d^*(1 \times a) = d^*(p_X^* a) = a$$

since the composition of $d : X \longrightarrow X \times X$ with the projection $p_X : X \times X \longrightarrow X$ is the identity.

Proposition 3.2. *The cross product determines the cup product and vice versa through the formulas*

$$
\begin{aligned}
a \smile b &= d^*(a \times b) \in H^*(X; R) \\
a \times b &= p_X^*(a) \smile p_Y^*(b) \in H^*(X \times Y; R).
\end{aligned}
$$

Proof. Plugging each formula into the other we verify

$$d^*(p_X^* a \smile p_Y^* b) = (d^* p_X^* a) \smile (d^* p_Y^* b) = a \smile b$$

on the one hand, and in the other direction

$$
\begin{aligned}
d^*\big(p_X^*(a) \times p_Y^*(b)\big) &= d^*(a \times 1 \times 1 \times b) \\
&= d^*(1 \times 1 \times a \times b) \\
&= (1 \times 1) \smile (a \times b) \\
&= 1 \smile (a \times b) = a \times b.
\end{aligned}
$$

using Lemma 3.1. Here the quadruple products are in $H^*(X \times Y \times X \times Y)$. □

Finally, we mention a more concrete construction leading to the explicit cup product formula in [Hat02]. Observe that, on the level of cochains, the cup product is dual to the composition

$$C_*(X) \xrightarrow{d_\#} C_*(X \times X) \xrightarrow{\theta} C_*(X) \otimes C_*(X).$$

Any chain map $C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$ which is natural in X and is the obvious map $x_0 \longmapsto x_0 \otimes x_0$ on 0-chains is called *diagonal approximation.* An acyclic models argument analogous to the proof of Proposition 1.6 shows

Proposition 3.3. *Any two diagonal approximations are chain homotopic.*

Let $\sigma : \Delta_n \longrightarrow X$ be an n-simplex, and identify Δ_n by an ordering of its vertices $[v_0, \dots, v_n]$. We define the *front p face* of σ to be the p-simplex

$$\mathrm{Fr}_p(\sigma) = \sigma|[v_0, \dots, v_p] : \Delta_p \longrightarrow X$$

and the *back q face* to be the q-simplex

$$\mathrm{Ba}_q(\sigma) = \sigma|[v_{n-q}, \dots, v_n] : \Delta_q \longrightarrow X.$$

Definition 3.4. The *Alexander-Whitney diagonal approximation* is the chain map $\mathrm{awd} : C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$ given on simplices by

$$C_n(X) \ni \sigma \longmapsto \sum_{p+q=n} \mathrm{Fr}_p(\sigma) \otimes \mathrm{Ba}_q(\sigma) \in \big(C_*(X) \otimes C_*(X)\big)_n$$

It can be checked that this is a well-defined chain map. It leads to Hatcher's explicit definition of the cup product, which coincides with the one defined here through Proposition 3.3.

References

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