MATH 540 EXAM 2 PRACTICE PROBLEM SOLUTIONS

Note: These are not guaranteed to be free of mistakes!

1. Theoretical exercises

Problem 1. Prove that if A is normal, then so are A^{T}, A^n for any n. Are the converses true? Can you find matrices such that A^T is normal but A is not, or such that A^2 is normal but A is not?

Solution. If A is normal, then

$$A^{\mathrm{T}}\left(A^*\right)^{\mathrm{T}} = \left(A^* A\right)^{\mathrm{T}} = \left(A\, A^*\right)^{\mathrm{T}} = \left(A^*\right)^{\mathrm{T}} A^{\mathrm{T}}$$

proving the first claim.

For the second, proceed by induction. The case $n = 1$ is true by assumption. Then

$$\begin{aligned} A^n \left(A^*\right)^n &= A^{n-1} A A^* \left(A^*\right)^{n-1} \\ &= A^{n-1} A^* A \left(A^*\right)^{n-1} \\ &= A^* A^{n-1} \left(A^*\right)^{n-1} A \\ &= A^* \left(A^*\right)^{n-1} A^{n-1} A \\ &= \left(A^*\right)^n A^n \end{aligned}$$

where on the third line we use normality iteratively $2n$ times to move the A^* to the beginning and the A to the end, and where we use the inductive hypothesis on the fourth line.

Since $\left(A^{\mathrm{T}}\right)^{\mathrm{T}} = A$, A is normal if and only if A^{T} is normal. On the other hand,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

is not normal, but $A^2 = I$ which is normal. $\square$

Problem 2. Prove that if T is unitary and self-adjoint, then the only eigenvalues of T can be ± 1. Do not assume the spectral theorem. Use the defining properties of unitarity and self-adjointness, along with properties of the determinant.

Solution. If T is unitary and self adjoint, then $T^2 = T^* T = I$. If $T\mathbf{x} = \lambda \mathbf{x}$, then

$$\lambda^2 \mathbf{x} = T^2 \mathbf{x} = \mathbf{x} \implies \lambda^2 = 1,$$

thus $\lambda = \pm 1$. (In fact, we didn't even need to use the determinant.) $\square$

Problem 3. Suppose P is self-adjoint, and all of its eigenvalues are either 0 or 1. Show P is a projection, which is the identity if 0 is not an eigenvalue.

Solution. Let k be the multiplicity of the eigenvalue 1. By the spectral theorem, there is an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ in which P has the block form

$$P = \begin{pmatrix} I_{k \times k} & 0 \\ 0 & 0 \end{pmatrix}$$

which is evidently a projection onto $E = \text{span}\,\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Equivalantly, there is a unitary transformation U such that

$$P = UDU^*$$

where $D = \text{d}iag\,\{1, \ldots, 1, 0, \ldots, 0\}$ is diagonal, and the columns of U are the vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. □

Problem 4. Let P be orthogonal projection onto a subspace E, and let Q be the projection onto $E^\perp$. Simplify the operator

$$9P^{83} + 3P^{12} + 13Q^{12} - Q$$

Solution. Since $P^2 = P$, therefore $P^n = P$ for all $n \geq 1$. Also, $Q = I - P$, so we have

$$\begin{aligned} 9P^{83} + 3P^{12} + 13Q^{12} - Q &= 9P + 3P + 13Q - Q \\ &= 12P + 12Q \\ &= 12P + 12(I - P) \\ &= 12I. \end{aligned}$$

□

Problem 5. A linear transformation $A : V \longrightarrow W$ is called *injective* if no two vectors can go to the same point; i.e. $A\mathbf{x}_1 = A\mathbf{x}_2 \iff \mathbf{x}_1 = \mathbf{x}_2$. A is called *surjective* if every vector in W is in the image of A: i.e. $\text{Ran}A = W$. Prove that

$$A \text{ is injective} \iff A^* \text{ is surjective.}$$

Solution. Injectivity is equivalent to the condition that $A\mathbf{x} = 0 \implies \mathbf{x} = 0$ using $A\mathbf{x}_1 = A\mathbf{x}_2 \iff A(\mathbf{x}_1 - \mathbf{x}_2) = 0$ and $\mathbf{x}_1 = \mathbf{x}_2 \iff \mathbf{x}_1 - \mathbf{x}_2 = 0$. Thus injectivity is equivalent to the requirement that $\text{Ker}\,(A) = \{0\}$.

Thus A is injective if and only if

$$\{0\} = \text{Ker}A = (\text{Ran}A^*)^\perp \iff \text{Ran}A = W$$

i.e. A^* is surjective. □

Problem 6. Use the trace to show that there cannot exist transformations T, S such that $TS - ST = I$. (Hint: think about eigenvalues).

Solution. OK, I guess eigenvalues actually have little to do with it. Anyway, recall that $\text{trace}\,(TS) = \text{trace}ST$. Thus

$$\text{trace}\,(TS - ST) = \text{trace}\,(TS) - \text{trace}\,(ST) = 0$$

while $\text{trace}I = n \neq 0$. Thus no such S, T can exist. □

Problem 7. Prove the following statements or find a counterexample:

(1) If $\mathbf{v}$ is an eigenvector of A then $\mathbf{v}$ is an eigenvector of A^n for any n.
(2) If $\mathbf{v}$ is an eigenvector of A, then $\overline{\mathbf{v}}$ is an eigenvector of A^*.
(3) If $\mathbf{v}$ is an eigenvector of A, then $\overline{\mathbf{v}}$ is an eigenvector of $\overline{A}$.

Solution. (1) is true since

$$A\mathbf{v} = \lambda\mathbf{v} \implies A^n\mathbf{v} = \lambda^n\mathbf{v}.$$

so $\mathbf{v}$ is an eigenvector of A^n with eigenvalue λ^n.

(3) is true by complex conjugation:

$$A\mathbf{v} = \lambda\mathbf{v} \implies \overline{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$$

so $\overline{\mathbf{v}}$ is an eigenvector of $\overline{A}$ with eigenvalue $\overline{\lambda}$.

On the other hand, (2) is false in general. One way to construct a counterexample is to use the spectral theorem. We may have $A^* = A$, for instance, which will be a counterexample provided there is an eigenvector $\mathbf{v}$ of A such that $\overline{\mathbf{v}}$ is not an eigenvector of A.

Thus we can take $A = UDU^*$, where U is a unitary matrix such that the complex conjugate of at least one of its columns does not appear among the others, say

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & 1 \end{pmatrix}$$

to pick a (random) example. Letting $D = \mathrm{d}iag\,\{1, 2\}$, for instance, we get

$$A = \frac{1}{2}\begin{pmatrix} 3 & 2-i \\ 2+i & 3 \end{pmatrix}$$

which has (if computations were done correctly) an eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

such that

$$\overline{\mathbf{v}} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is not an eigenvector of $A^* = A$. □

Problem 8. Suppose $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of an operator A. Let $\alpha \in \mathbb{C}$. What are the eigenvalues of $A + \alpha I$?

Solution. From the characterization of eigenvalues as roots of the characteristic polynomial, we have

$$p_{A+\alpha I}(\lambda) = \det\left((A + \alpha I) - \lambda I\right) = \det\left(A - (\lambda - \alpha)I\right) = p_A(\lambda - \alpha).$$

Equivalently, $p_{A+\alpha I}(\lambda + \alpha) = p_A(\lambda)$ so $\{\lambda_1 + \alpha, \ldots, \lambda_n + \alpha\}$ are the eigenvalues of $A + \alpha I$. □

Problem 9. Let $M_{2\times 2}$ be the space of 2×2 real matrices. For any $A \in M_{2\times 2}$, let $T_A : M_{2\times 2} \longrightarrow M_{2\times 2}$ be the operator given by

$$T_A X = AX$$

(1) Let U be an invertible matrix. Show that if $X \in M_{2\times 2}$ is an eigenvector of T_A, then UX is an eigenvector of $T_{UAU^{-1}}$ with the same eigenvalue.
(2) Show that the eigenvalues of $T_A : M_{2\times 2} \longrightarrow M_{2\times 2}$ coincide with those of $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. What happens to the multiplicities? (Hints: show it directly, or use the first part of the problem to reduce to the case that A is upper triangular.)

Solution. For part (1), suppose $T_A X = \lambda X$. Then

$$T_{UAU^{-1}}UX = UAU^{-1}UX = UAX = U\lambda X = \lambda UX$$

so UX is an eigenvector of $T_{UAU^{-1}}$ with eigenvalue λ.

For part (2), here is a direct proof. Suppose λ is an eigenvalue of T_A, so $T_A X = \lambda X$ for some X. Then $AX = \lambda X$ which says that λ is an eigenvalue of A, and the columns of X are eigenvectors. On the other hand, if $A\mathbf{v} = \lambda\mathbf{v}$, then the matrix X both of whose columns are $\mathbf{v}$ is an eigenvector of T_A with eigenvalue λ.

Alternatively, it can be shown directly. The computation is made easier by using a Schur decomposition $A = UBU^*$ for A, with upper triangular matrix

$$B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

so A has eigenvalues $a, c \in \mathbb{C}$. As computed on the previous exam, using the obvious basis for 2×2 matrices, T_B is represented by the 4×4 matrix

$$T_B = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

which is also upper triangular, and therefore has eigenvalues $a, c \in \mathbb{C}$ but with multiplicity 2. By part (1), T_B has the same eigenvalues as $T_{UBU^{-1}}$.

Thus T_A has the same eigenvalues as A, but with double the multiplicity. □

2. Computational exercises

Problem 10. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

That is, find matrices $P$and D such that P is orthogonal (so $P^{-1} = P^{\mathrm{T}}$), D is diagonal, and $A = PDP^{\mathrm{T}}$.

Solution. A is self-adjoint, hence it has real eigenvalues, so this will be possible. Indeed,

$$p(\lambda) = A - \lambda I = \lambda^2 - 1$$

so A has eigenvalues $\{1, -1\}$. For $\lambda = 1$, an obvious eigenvector is

$$\begin{pmatrix} \sin\theta \\ 1 - \cos\theta \end{pmatrix}$$

but its norm is $\sqrt{2 - 2\cos\theta}$, so we take

$$\mathbf{v}_1 = \frac{1}{\sqrt{2 - 2\cos\theta}} \begin{pmatrix} \sin\theta \\ 1 - \cos\theta \end{pmatrix}$$

which is normalized.

For $\lambda = -1$, we can similarly take

$$\mathbf{v}_2 = \frac{1}{\sqrt{2 + 2\cos\theta}} \begin{pmatrix} -\sin\theta \\ 1 + \cos\theta \end{pmatrix}.$$

Note that $\mathbf{v}_1 \perp \mathbf{v}_2$, which is guaranteed since A is self-adjoint and the eigenvalues are distinct, but this is a good check to make sure we haven't made mistakes.

In any case, we conclude that $A = PDP^{\mathrm{T}}$, where

$$P = \begin{pmatrix} \frac{\sin\theta}{\sqrt{2-2\cos\theta}} & \frac{-\sin\theta}{\sqrt{2+2\cos\theta}} \\ \frac{1-\cos\theta}{\sqrt{2-2\cos\theta}} & \frac{1+\cos\theta}{\sqrt{2+2\cos\theta}} \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

Problem 11. Find the best fit linear curve $y = ax+b$ for the data points $(x_n, y_n) \in \{(1,1), (-2,1), (3,4), (2,3)\}$.

Solution. The corresponding linear problem is $A\mathbf{v} = \mathbf{w}$, where

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \\ 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 3 \end{pmatrix}$$

which of course has no solution. The least squares solution is given by the normal equation

$$A^*A\mathbf{v} = A^*\mathbf{w}$$

where

$$A^*A = \begin{pmatrix} 18 & 4 \\ 4 & 4 \end{pmatrix}, \quad A^*\mathbf{w} = \begin{pmatrix} 17 \\ 9 \end{pmatrix}.$$

A^*A is invertible, with

$$(A^*A)^{-1} = \frac{1}{64}\begin{pmatrix} 4 & -4 \\ -4 & 18 \end{pmatrix}$$

and the solution is given by

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1/2 \\ 47/32 \end{pmatrix}.$$

□

Problem 12. Compute the following determinants:

(1)

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix}$$

(2)

$$\begin{vmatrix} 2 & 4 & 9 & 6 \\ 0 & 2 & 4 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & -2 & -4 & 1 \end{vmatrix}$$

Solution. For the first matrix,

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 24.$$

For the second,

$$\begin{vmatrix} 2 & 4 & 9 & 6 \\ 0 & 2 & 4 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & -2 & -4 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 4 & 1 \\ 2 & 4 & 9 & 6 \\ 0 & -2 & -4 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & -2 & -4 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -12$$

where we did the following sequence of moves $R_1 \leftrightarrow R_3$, $R_3 \longmapsto R_3 - 2R_1$, and $R_4 \longmapsto R_4 + R_2$.

□

Problem 13. Diagonalize the matrix

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}$$

and find all possible square roots; i.e. complex 2×2 matrices B such that $B^2 = A$.

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3)$$

so the eigenvalues are $\{3, -2\}$. Corresponding normalized eigenvectors are given by

$$\lambda = 3 : \mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda = -2 : \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Thus

$$A = UDU^* = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

There are four possible square roots, $B = US_iU^*$, $i = 1, 2, 3, 4$, where

$$S_1 = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2}\, i \end{pmatrix}, \qquad S_2 = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{2}\, i \end{pmatrix},$$

$$S_3 = \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{2}\, i \end{pmatrix}, \qquad S_4 = \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & -\sqrt{2}\, i \end{pmatrix}.$$

□

Problem 14. Compute the projections onto the four fundamental subspaces Ran$A$, Ker$A$, RanA^*, and KerA^*, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 6 \end{pmatrix}$$

Solution. Row reduction leads to the echelon form

$$A_e = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the first two columns of A, $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{(1,2,3)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}\right\}$ form a basis for Ran$\,(A)$, and we can also solve for the kernel, getting

$$\mathrm{Ker}\,(A) = \mathrm{span}\,\{\mathbf{w}\}, \quad \mathbf{w} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

Projection onto Ker$\,(A)$ is given by the matrix

$$P_{\mathrm{Ker}(A)} = \frac{1}{\|\mathbf{w}\|^2} \mathbf{w}\mathbf{w}^* = \frac{1}{9} \begin{pmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

Our basis for Ran$\,(A)$ is not orthogonal, so we need to apply Gram-Schmidt. It will involve less calculation to take $\mathbf{x}_1 = \mathbf{v}_2 = (0,1,0)^{\mathrm{T}}$ (which is already normalized), since projection onto this subspace is easy. So then

$$\mathbf{x}_2 = \mathbf{v}_1 - (\mathbf{v}_1, \mathbf{x}_1)\, \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

(Note that we would have obtained this directly by taking the second two columns of A instead of the first two, if we had seen that they were independent and that the first was a linear combination of them.) In any case, we have

$$P_{\mathrm{Ran}(A)} = \frac{1}{\|\mathbf{x}_1\|^2}\mathbf{x}_1\mathbf{x}_1^* + \frac{1}{\|\mathbf{x}_2\|^2}\mathbf{x}_2\mathbf{x}_2^* = \frac{1}{10}\begin{pmatrix}1 & 0 & 3\\ 0 & 10 & 0\\ 3 & 0 & 9\end{pmatrix}.$$

By the fact that $\mathrm{Ran}\,(A^*) = (\ker(A))^{\perp}$, we compute

$$P_{\mathrm{Ran}(A^*)} = I - P_{\mathrm{Ker}(A)} = \frac{1}{9}\begin{pmatrix}5 & 4 & 2\\ 4 & 5 & -2\\ 2 & -2 & 8\end{pmatrix}$$

Similarly,

$$P_{\mathrm{Ker}(A^*)} = I - P_{\mathrm{Ran}(A)} = \frac{1}{10}\begin{pmatrix}9 & 0 & -3\\ 0 & 0 & 0\\ -3 & 0 & 1\end{pmatrix}.$$ □

Problem 15. Which of the following pairs of matrices are similar? Which are unitarily equivalent?

(1) $\begin{pmatrix}1 & 3\\ 4 & 2\end{pmatrix}$ and $\begin{pmatrix}0 & 2\\ 2 & 3\end{pmatrix}$

(2) $\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$

(3) $\begin{pmatrix}1 & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & i\end{pmatrix}$ and $\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}$

Solution. The matrices in (1) cannot be similar since their determinants are different (which means their eigenvalues cannot agree and therefore they cannot be similar).

In (2), both matrices have the same eigenvalues, namely $\lambda = 1$ and $\lambda = 2$ which are distinct, and therefore both matrices have a basis of eigenvectors, i.e. are diagonalizable. Therefore they are similar. On the other hand, they cannot be unitarily equivalent since the right hand matrix is self adjoint, while the left hand one is not, and unitary equivalence preserves self-adjointness.

In (3), we compute the eigenvalues of the right hand matrix to be $\{1, i, -i\}$, which are also (obviously) the eigenvalues of the left hand matrix. One can either verify that the eigenvectors of the right matrix are orthogonal, or (which is much easier) check that it is normal, and hence unitarily diagonalizable. Since it shares eigenvalues with the left matrix, they must be unitarily equivalent (hence also similar). □