Math 350 Problem Set 8 Solutions

Part I

1. Find the values of $a \geq 0$ such that the following integrals exist. Justify your answers

   (a) (6pts) $D_1 = \{(x,y) \mid x^2+y^2 \leq 1\}$

$$I_a = \iint_{D_1} \frac{1}{(x^2+y^2)^a}\, dA$$

Solution. By Fubini's Theorem, since $f(x,y) = (x^2+y^2)^{-a} \geq 0$, I_a will exist if it has a finite value as an iterated, improper integral. Furthermore, we can use a change of variables, since this does not change the property that $f \geq 0$. So we look at the improper, iterated integral

$$\int_0^{2\pi}\int_0^1 \frac{1}{r^{2a}} r\, dr\, d\theta = \begin{cases} 2\pi r^{-2a+2}\Big|_{r=0}^{r=1} & \text{if } a \neq 1, \\ 2\pi \ln r\Big|_{r=0}^{r=1} & \text{if } a = 1. \end{cases}$$

This is finite provided $a < 1$, so for $a < 1$, I_a exists.

That it does not exist for $a \geq 1$ follows by integrating $f(x,y)$ over the annulus $D_{1,\epsilon} = \{(x,y) \mid \epsilon \leq x^2+y^2 \leq 1\}$, and taking $\epsilon \to 0$:

$$I_{a,\epsilon} = \iint_{D_{1,\epsilon}} \frac{1}{(x^2+y^2)^a}\, dA = \int_0^{2\pi}\int_\epsilon^1 \frac{1}{r^{2a}} r\, dr\, d\theta = \begin{cases} 2\pi(1-\epsilon^{-2a+2}) & \text{if } a > 1, \\ 2\pi \ln 1/\epsilon & \text{if } a = 1. \end{cases}$$

both of which diverge as $\epsilon \to 0$.

   (b) (6pts) $D = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y \leq x\}$,

$$I_a = \iint_{D_2} \frac{1}{(x-y)^a}\, dA$$

Solution. As with the previous problem, since $f \geq 0$, it suffices to check integrability as an iterated integral, using Fubini's theorem. For $a \neq 1$, $a \neq 2$, we have

$$\begin{aligned} I_{a,\epsilon,\delta,\sigma} &= \int_\delta^1 \int_\sigma^{x-\epsilon} \frac{1}{(x-y)^a}\, dy\, dx \\ &= \int_\delta^1 \frac{(x-\sigma)^1 - a}{1-a} - \frac{\epsilon^{1-a}}{1-a}\, dx \\ &= \frac{(1-\sigma)^{2-a}}{(1-a)(2-a)} - \frac{(\delta-\sigma)^{2-a}}{(1-a)(2-a)} - (1-\delta)\frac{\epsilon^{1-a}}{(1-a)} \end{aligned}$$

The limit as ϵ, δ and σ go to 0 will exist provided $a < 1$. (Note that this implies $a < 2$ we don't have to worry about the $\delta - \sigma$ term.)

For $a = 1$, we have

$$I_{a,\epsilon,\delta,\sigma} = \int_\delta^1 \int_\sigma^{x-\epsilon} \frac{1}{(x-y)}\, dy\, dx = \int_\delta^1 \ln(x-\sigma) - \ln(\epsilon)\, dx$$

which will certainly not be finite as $\epsilon \to 0$.

For $a = 2$, we have

$$I_{a,\epsilon,\delta,\sigma} = \int_\delta^1 \int_\sigma^{x-\epsilon} \frac{1}{(x-y)^2}\, dy\, dx = \int_\delta^1 \epsilon^{-1} - (x-\sigma)^{-1}\, dx$$

which will also fail to be finite as $\epsilon \to 0$.

We conclude that the integral exists for all $a < 1$.

2. (6pts) Show that if $\mathcal{C}$ is a curve defined in polar coordinates by $r = r(\theta), \theta_0 \le \theta \le \theta_1$, then the path integral of $f(x,y)$ over $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} f\,ds = \int_{\theta_0}^{\theta_1} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \frac{dr}{d\theta}^2}\,d\theta.$$

(Hint: how is this curve parametrized? Don't let yourself get confused just because the parametrizing variable is a different one than you're used to!)

Solution. We use $\mathbf{c} : \theta \mapsto (r(\theta)\cos\theta, r(\theta)\sin\theta)$, $\theta_0 \le \theta \le \theta_1$ as the parametrization of $\mathcal{C}$. Using the arclength formula

$$\begin{aligned} ds = \|\mathbf{c}'(\theta)\|\,d\theta &= \sqrt{(r'(\theta)\cos\theta + r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2}\,d\theta \\ &= \sqrt{r'(\theta)^2\left(\cos^2\theta + \sin^2\theta\right) + r(\theta)^2\left(\cos^2\theta + \sin^2\theta\right) \pm r'(\theta)r(\theta)\cos\theta\sin\theta\,d\theta} \end{aligned}$$

yeilds the result.

3. (6pts) Consider the spherical surface $\rho = a$ where $a \in \mathbb{R}$ is constant. Show that, in terms of variables (ϕ, θ),

$$dS = a^2\sin\phi\,d\phi\,d\theta$$

on this surface.

Solution. We use $(\phi,\theta) \mapsto (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$ as our parametrization. We have

$$\mathbf{T}_\phi = a\cos\phi\cos\theta\mathbf{i} + a\cos\phi\sin\theta\mathbf{j} - a\sin\phi\mathbf{k}$$

and

$$\mathbf{T}_\theta = -a\sin\phi\sin\theta\mathbf{i} + a\sin\phi\cos\theta\mathbf{j}.$$

So

$$\begin{aligned} \mathbf{T}_\phi \times \mathbf{T}_\theta = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{bmatrix} \\ = a^2\left(\sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + (\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta)\mathbf{k}\right). \end{aligned}$$

Taking the length of this vector, we obtain

$$dS = a^2\sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \cos^2\phi\sin^2\phi\,d\phi\,d\theta} = a^2\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi\,d\phi\,d\theta} = a^2\sin\phi\,d\phi\,d\theta.$$

4. (6pts) Consider the surface $\phi = \alpha$ where $\alpha \in [0,\pi]$ is constant. What does this surface look like? Show that, in terms of variables (ρ, θ),

$$dS = \rho\sin\alpha\,d\rho\,d\theta$$

(Note that there is only a single power of ρ!).

Solution. We use $(\rho,\theta) \mapsto (\rho\sin\alpha\cos\theta, \rho\sin\alpha\sin\theta, \rho\cos\alpha)$ as our parametrization. We have

$$\mathbf{T}_\rho = \sin\alpha\cos\theta\mathbf{i} + \sin\alpha\sin\theta\mathbf{j} + \cos\alpha\mathbf{k}$$

and

$$\mathbf{T}_\theta = -\rho\sin\alpha\sin\theta\mathbf{i} + \rho\sin\alpha\cos\theta\mathbf{j}.$$

So

$$\mathbf{T}_\rho \times \mathbf{T}_\theta = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\alpha\cos\theta & \cos\alpha\sin\theta & -\sin\alpha \\ -\rho\sin\alpha\sin\theta & \rho\sin\alpha\cos\theta & 0 \end{bmatrix}$$
$$= \rho\left(\sin^2\alpha\cos\theta\mathbf{i} + \sin^2\alpha\sin\theta\mathbf{j} + (\cos\alpha\sin\alpha\cos^2\theta + \cos\alpha\sin\alpha\sin^2\theta)\mathbf{k}\right).$$

Taking the length of this vector, we obtain

$$dS = \rho\sqrt{\cos^2\theta\sin^4\alpha + \sin^2\theta\sin^4\alpha + \cos^2\alpha\sin^2\alpha}\,d\rho\,d\theta = \rho\sqrt{\sin^4\alpha + \cos^2\alpha\sin^2\alpha}\,d\rho\,d\theta = \rho\sin\alpha\,d\rho\,d\theta.$$

5. (10pts) On a surface defined by $z = g(x, y)$, our formula for the oriented surface area element is

$$\mathbf{n}\,dS = d\mathbf{S} = (-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})\ dx\,dy,$$

where $g_x = \frac{\partial g}{\partial x}(x, y)$ and $g_y = \frac{\partial g}{\partial y}(x, y)$. Use this to show that if the same surface is also defined by the equation $f(x, y, z) = c$ for some constant c, then

$$\mathbf{n}\,dS = d\mathbf{S} = \frac{\nabla f}{f_z}\,dx\,dy.$$

Solution. Parametrize the surface using $(x, y) \mapsto (x, y, g(x, y))$, and let $h(x, y)$ be $f(x, y, z)$ restricted to the surface, so

$$h(x, y) = f(x, y, g(x, y))$$

Since, by definition, $f = c$ is a constant function on the surface, we must have

$$\nabla h(x, y) = (f_x(x, y, g(x, y)) + f_z(x, y, g(x, y))g_x(x, y))\,\mathbf{i} + (f_y(x, y, g(x, y)) + f_z(x, y, g(x, y))g_y(x, y))\,\mathbf{j} = \mathbf{0}$$

So in particular,

$$-g_x = \frac{f_x}{f_z} \qquad -g_y = \frac{f_y}{f_z}$$

on the surface. Then

$$dS = (-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})\ dx\,dy = \left(\frac{f_x}{f_z}\mathbf{i} + \frac{f_y}{f_z}\mathbf{j} + \frac{f_z}{f_z}\mathbf{k}\right)\ dx\,dy = \frac{\nabla f}{f_z}\,dx\,dy.$$

Part II

1. Let $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, and $\mathcal{C}$ be the oriented curve defined by $(\cos t, \sin t, t)$ for $0 \le t \le 2\pi$. Let

$$I = \int_{\mathcal{C}} \mathbf{F}\cdot\,d\mathbf{s}.$$

(a) (3pts) Evaluate I directly.
Solution. We have $\mathbf{F}(x(t), y(t), z(t)) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2t\mathbf{k}$, and $d\mathbf{s} = (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})\ dt$ so

$$I = \int_{\mathcal{C}} \mathbf{F}\cdot\,d\mathbf{s} = \int_0^{2\pi} -\sin^2 t + \cos^2 t + 2t\,dt = \int_0^{2\pi} \cos 2t + 2t\,dt = 4\pi^2.$$

(b) (3pts) $\mathbf{F}(x, y, z)$ is in fact a conservative vector field. Evaluate I by finding a potential function f such that $\mathbf{F} = \nabla f$ and use the fundamental theorem of calculus for line integrals.

Solution. We need

$$f_x(x,y,z) = y \implies f(x,y,z) = xy + g(y,z).$$

Then we have

$$f_y(x,y,z) = x + g_y(y,z) = x \implies g_y(y,z) = 0 \implies g(y,z) = g(z).$$

Finally,

$$f_z(x,y,z) = g_z(z) = 2z \implies g(z) = z^2 + c$$

so we can take

$$f(x,y,z) = xy + z^2.$$

Then

$$I = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} = f(1,0,2\pi) - f(1,0,0) = 4\pi^2.$$

2. (6pts) Compute the path integral

$$\int_{\mathbf{c}} \frac{(x+y)}{(y+z)}\, ds, \quad \mathbf{c}(t) = (t, 2/3t^{3/2}, t),\ 1 \le t \le 2$$

Solution. In terms of our parametrization, we have

$$ds = \|\mathbf{c}'(t)\|\, dt = \sqrt{2+t}\, dt,$$

and

$$\frac{(x+y)}{(y+z)} = \frac{(t+2/3t^{3/2})}{(2/3t^{3/2}+t)} = 1.$$

So

$$\int_{\mathbf{c}} \frac{(x+y)}{(y+z)}\, ds = \int_1^2 \sqrt{2+t}\, dt = \frac{2}{3}(2+t)^{3/2}\Big|_{t=1}^{t=2} = \frac{16}{3} - 2\sqrt{3}.$$

3. (6pts) Compute the line integral

$$\int_{\mathbf{c}} x\, dy - y\, dx, \quad \mathbf{c}(t) = (\cos t, \sin t),\ 0 \le t \le \pi/2$$

(You may want to write it in the form $\mathbf{F} \cdot d\mathbf{s}$ first if that helps you.)
Solution. In terms of our parametrization,

$$x\, dy - y\, dx = \cos t(\cos t\, dt) - \sin t(-\sin t\, dt) = 1\, dt,$$

so

$$\int_{\mathbf{c}} x\, dy - y\, dx = \int_0^{\pi/2} 1\, dt = \pi/2.$$

4. Find the surface area of the following surfaces:

(a) (4pts) $z = 4 - x^2 - y^2$, $z \ge 0$.
Solution. Using

$$dS = \sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy = \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy,$$

we have

$$\iint_S dS = \iint_R \sqrt{1 + 4(x^2+y^2)}\, dx\, dy$$

where $R = \{(x,y) \mid x^2 + y^2 \le 4\}$. Thus,

$$\iint_S dS = \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2}\, r\, dr\, d\theta = 2\pi \frac{2}{24}\left(1+4r^2\right)^{3/2}\Big|_{r=0}^{r=2} = \frac{\pi}{6}\left(65^{3/2} - 1\right).$$

(b) (4pts) The sphere of radius a (Hint: you may want to use the result from problem 3 in Part I)
Solution. Using the surface area formula from problem 3, we have

$$\iint_{\mathcal{S}_a} dS = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta = 4\pi a^2.$$

(c) (4pts) The cone $z = \sqrt{x^2 + y^2}$, $z \leq 1$. (Hint: problem 4 in Part I gives one possibility)
Solution. This is the cone $z = r$, which can be parametrized using spherical coordindates ρ and θ with $\phi = \pi/4$ fixed. i Using the formula from problem 4, we have

$$\iint_{\mathcal{S}} dS = \int_0^{2\pi} \int_0^{1} \rho \sin\pi/4 \, d\rho \, d\theta = 2\pi \frac{\sqrt{2}}{2} = \sqrt{2}\pi.$$

(Plus an additional π if you include the top circular "cap" on the cone. Either way is fine.)

5. (4pts) Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the oriented surface consisting of the upper unit hemisphere, with upward pointing normal.

Solution. Using

$$dS = a^2 \sin\phi \, d\phi \, d\theta = \sin\phi \, d\phi \, d\theta$$

and

$$\hat{\mathbf{n}} = \frac{1}{a}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

we have

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iint_{\mathcal{S}} x^2 + y^2 + z^2 \, dS = \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \, d\phi \, d\theta = 2\pi.$$