Math 350 Problem Set 5 Solutions

Part I

1. (10pts) Show that
$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy \right) dx = \frac{\pi}{4}$$
while
$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx \right) dy = -\frac{\pi}{4}.$$
You may use the fact that
$$\int_0^1 \frac{s^2 - t^2}{(s^2+t^2)^2}\, dt = \frac{1}{1+s^2}.$$
Why doesn't this violate either version of Fubini's theorem (Theorem 3 or 3')?

Proof. The identity above is obtained as follows (you did not need to show this):
$$\begin{aligned}
\int_0^1 \frac{s^2 - t^2}{(s^2+t^2)^2}\, dt &= \int_0^1 \frac{s^2 + t^2 - 2t^2}{(s^2+t^2)^2}\, dt \\
&= \int_0^1 \frac{1}{s^2+t^2}\, dt + \int_0^1 t \cdot \frac{(-2t)}{(s^2+t^2)^2}\, dt \\
&= \int_0^1 \frac{1}{s^2+t^2} + \left.\frac{t}{s^2+t^2}\right|_{t=0}^1 - \int_0^1 \frac{1}{s^2+t^2}\, dt \\
&= \frac{1}{s^2+1}.
\end{aligned}$$
using integration by parts, where
$$\frac{-2t}{(s^2+t^2)^2} = \frac{d}{dt}\left(\frac{1}{s^2+t^2}\right).$$
Interchanging s and t, we see that
$$\int_0^1 \frac{s^2 - t^2}{(s^2+t^2)^2}\, ds = -\int_0^1 \frac{t^2 - s^2}{(s^2+t^2)^2}\, ds = -\frac{1}{1+t^2}.$$
Thus,
$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy \right) dx = \int_0^1 \frac{1}{1+x^2}\, dx = \arctan x \Big|_{x=0}^1 = \frac{\pi}{4}$$
whereas
$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx \right) dy = \int_0^1 -\frac{1}{1+y^2}\, dy = -\arctan y \Big|_{y=0}^1 = -\frac{\pi}{4}.$$
This does not violate either version of Fubini's theorem since the integrand is neither continuous nor bounded near $(0, 0)$. $\square$

2. (10pts) Let $A \subset \mathbb{R}^2$. Suppose $f(x,y)$ is continuous and non-negative: $f(x,y) \geq 0$. Prove that if $\iint_A f(x,y)\,dA = 0$, then $f(x,y) = 0$ for all $(x,y) \in A$.

Proof. Assume $f(x_0, y_0) = c > 0$. We will show that $\iint_A f\,dA \neq 0$ (which is the contrapositive of the statement we're trying to prove, hence equivalent). By continuity of f, there exists a $\delta > 0$ such that

$$\|(x,y) - (x_0,y_0)\| < \delta \implies |f(x,y) - f(x_0,y_0)| < \frac{c}{2} \implies f(x,y) > \frac{c}{2}.$$

Since

$$|x - x_0| < \delta/\sqrt{2} \text{ and } |y - y_0| < \delta/\sqrt{2} \implies \|(x,y) - (x_0,y_0)\| < \delta,$$

the disk $\|(x,y) - (x_0,y_0)\| < \delta$ contains a rectangle R of area $2\delta^2 = \left(\frac{2\delta}{\sqrt{2}}\right)\left(\frac{2\delta}{\sqrt{2}}\right)$. Let $B = A \setminus R$ be the region obtained by deleting R from A. By additivity and monotonicity ($f \geq 0$ and $f \geq \frac{c}{2}$ on R),

$$\iint_A f\,dA = \iint_B f\,dA + \iint_R f\,dA \geq 0 + (2\delta)\frac{c}{2} > 0.$$

and therefore $\iint_A f\,dA \neq 0$.

Assuming we know additivity for more general regions, we could alternatively just let R be the disk $\|(x,y) - (x_0,y_0)\| < \delta$. □

3. (10pts) Let $R = [0,1] \times [0,1]$ and let $f : R \to \mathbb{R}$ be the function

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is *not* integrable, by showing that the sequence of Riemann sums does not tend to a unique limit which is independent of the choice of points $\mathbf{c}_{jk}$.

We will produce two convergent sequences of Riemann approximations, which converge to different values. Let

$$S_n = \sum_{i,j=1}^{n} f(\mathbf{c}_{ij})\Delta x \Delta y$$

where $\mathbf{c}_{ij} = (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and x and y are rational. On the other hand, let

$$S'_n = \sum_{i,j=1}^{n} f(\mathbf{c}'_{ij})\Delta x \Delta y$$

where $\mathbf{c}'_{ij} = (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and at least one of x or y is irrational. This is possible since there are both rational and irrational points in any interval $[a,b]$ as long as $b > a$.

Thus

$$S_n = \sum_{i,j=1}^{n} 1\,\Delta x\,\Delta y = \sum_{i,j=1}^{n} \frac{1}{n^2} = \frac{n^2}{n^2} = 1$$

and

$$S'_n = \sum_{i,j=1}^{n} 0\,\Delta x\,\Delta y = 0.$$

Doing this for every n, we obtain sequences

$$\{S_n\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty} \quad \text{and} \quad \{S'_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$$

both of which converge, but

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 = 1 \neq \lim_{n\to\infty} S'_n = \lim_{n\to\infty} 0 = 0.$$

Thus f is not integrable.