Math 2420 – Problem Set 3, due Monday 3/12.

Update 3/12: There is some sign involved in the cap/cross product formula in problem 3. Thanks to Yilong for pointing this out.

Problem 1. Show that manifolds M and N are orientable if and only if $M \times N$ is orientable. Show also that if $\alpha \in H_m(M;R)$ and $\beta \in H_n(N;R)$ are fundamental classes, then $\alpha \times \beta \in H_{m+n}(M \times N;R)$ is a fundamental class.

Solution. Note that the cross product forms a map

$$\times : H_m(M \,|\, x;R) \otimes H_n(N \,|\, y;R) \longrightarrow H_{m+n}(M \times N \,|\, x \times y;R) \tag{1}$$

since $(M \setminus x) \times N \cup M \times (N \setminus y) = M \times N \setminus x \times y$. Suppose for a moment that $R = \mathbb{Z}$. From the Künneth theorem (of which a fully relative version exists when the second factor in each pair is an open set)

$$\begin{aligned} H_{m+n}(M \times N \,|\, x \times y) &\cong \bigoplus_{p+q=n+m} H_p(M \,|\, x) \otimes H_q(N \,|\, y) \oplus \bigoplus_{p+q=m+n-1} \operatorname{Tor}\big(H_p(M \,|\, x), H_q(N \,|\, y)\big) \\ &\equiv H_m(M \,|\, x) \otimes H_n(N \,|\, y) \end{aligned}$$

since only $H_m(M \,|\, x)$ and $H_n(N \,|\, y)$ are nonzero. Thus (1) is an isomorphism for $R = \mathbb{Z}$. In the case of an arbitrary commutative ring R with identity it still holds that the product of generators is a generator (i.e. $\pm 1 \times 1 = \pm 1$ still holds in (1)). We obtain analogous results by replacing x and y by closed balls of finite radius.

If $x \longmapsto \mu_x \in H_m(M \,|\, x;R)$ and $y \longmapsto \nu_y \in H_n(N \,|\, y;R)$ are orientations for M and N it follows that

$$(x, y) \longmapsto \mu_x \times \nu_y \in H_{m+n}(M \times N \,|\, x \times y;R)$$

is an orientation for $M \times N$ (it satisfies the compatibility condition by passing to the product of closed balls $B_1 \ni x$ and $B_2 \ni y$.) The converse is similar.

For the result concerning fundamental classes, first consider $R = \mathbb{Z}$ again. From the Künneth theorem

$$\begin{aligned} H_{m+n}(M \times N) &\cong H_m(M) \otimes H_n(N) \oplus \operatorname{Tor}\big(H_m(M), H_{n-1}(N)\big) \oplus \operatorname{Tor}\big(H_{m-1}(M), H_n(N)\big) \\ &= H_m(M) \otimes H_n(N) \end{aligned}$$

since all other terms vanish and the Tor groups vanish since the top degree groups are free. Again this implies that, for a general ring R, the map

$$\times : H_m(M;R) \otimes H_n(N;R) \longrightarrow H_{m+n}(M \times N;R)$$

sends the product of generators to a generator. Since fundamental classes just amount to a choice of generators for these groups, the result follows. Alternatively, one can use that the restriction maps $\alpha \longmapsto \alpha_x$ induced by $M \longrightarrow (M, M \setminus x)$ and similarly for β satisfy

$$(\alpha \times \beta)_{(x,y)} = \alpha_x \times \beta_y$$

by the earlier isomorphism. $\square$

Problem 2. For a (locally compact Hausdorff) space X let X^+ denote the one-point compactification. If the added point $\infty \in X^+$ has a neighborhood which is a cone with ∞ as a cone point (a neighborhood deformation retract) show that the evident map

$$H_c^n(X; G) \longrightarrow H^n(X^+, \infty; G)$$

is an isomorphism for all n.

Solution. Recall that the compact Hausdorff space $X^+ = X \cup \{\infty\}$ is equipped with a topology such that for any open set $V \subset X^+$ with $\infty \in V$, the complement $X \setminus V$ is compact in X.

Now consider any compact set $K \subset X$. Since X^+ is Hausdorff, there are open neighborhoods U of K and V of ∞ in X^+ which are disjoint. By the assumption that ∞ has a neighborhood deformation retract, we may assume (by making V smaller if necessary) that V itself deformation retracts to ∞. Letting $L = X \setminus V$, it follows that L is compact, $K \subset L$ and furthermore for all n,

$$H^n(X \mid L) = H^n(X, X \setminus L) = H^n(X, V) \cong H^n(X^+, V) \cong H^n(X^+, \infty),$$

where we have used excision and homotopy equivalence to obtain the two isomorphisms.

Thus we have demonstrated that for every compact $K \subset X$ there is a compact set L such that for all n, there is a homomorphism $H^n(X \mid K) \longrightarrow H^n(X \mid L)$ (induced by the inclusion $K \subset L$) and $H^n(X \mid L) \cong H^n(X^+, \infty)$. Thus in the computation of cohomology with compact supports, the direct limit may be taken just over these sets $L \subset X$ (this was a property of direct limits mentioned in class). To wit,

$$H_c^n(X; G) = \varinjlim_{K \subset X} H^n(X \mid K; G) = \varinjlim_{L \subset X} H^n(X \mid L; G) \cong H^n(X^+, \infty; G)$$

for each n. $\square$

Problem 3. Prove that, for $\phi \in H^*(X)$, $\psi \in H^*(Y)$, $a \in H_*(X)$ and $b \in H_*(Y)$, the cap and cross products are related by

$$(a \times b) \frown (\phi \times \psi) = (-1)^*(a \frown \phi) \times (b \frown \psi),$$

with a sign that you should determine. Use this to compute all cap products in homology and cohomology of $S^m \times S^n$, where m and n may be equal.

Solution. The easiest way to determine the sign is the following argument. Let $\langle , \rangle : H^n(X) \times H_n(X) \longrightarrow \mathbb{Z}$ be the pairing between cohomology and homology (or the cap product, if you like, where the degrees are equal). Recall the following properties:

$$\langle \alpha, a \frown \beta \rangle = \langle \beta \smile \alpha, a \rangle, \quad |a| = |\alpha| + |\beta| .$$
$$\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \langle \beta, b \rangle, \quad |a| = |\alpha|, \ |b| = |\beta| ,$$

where the notation $|\cdot|$ denotes the degree.

In addition to ϕ, ψ, a and b as above, suppose there are also given classes $\alpha \in H^*(X)$, $\beta \in H^*(Y)$ such that

$$|a| = |\alpha| + |\phi| , \quad |b| = |\beta| + |\psi| .$$

Then we may compute

$$\begin{aligned} \langle \alpha \times \beta, (a \times b) \frown (\phi \times \psi) \rangle &= \langle (\phi \times \psi) \smile (\alpha \times \beta) , a \times b \rangle \\ &= (-1)^{|\psi||\alpha|} \langle (\phi \smile \alpha) \times (\psi \smile \beta) , a \times b \rangle \\ &= (-1)^{|\psi||\alpha|} \langle \alpha, a \frown \phi \rangle \langle \beta, b \frown \psi \rangle \\ &= (-1)^{|\psi||\alpha|} \langle \alpha \times \beta, (a \frown \phi) \times (b \frown \psi) \rangle \end{aligned}$$

The second line follows from the formulas for the cup product and cross product in terms of one another and graded commutativity. Thus in order for the formula to hold, the sign must be given by

$$(a \times b) \frown (\phi \times \psi) = (-1)^{|\psi|(|a|+|\phi|)}(a \frown \phi) \times (b \frown \psi).$$

Note that, if every element of $H^*(X \times Y)$ was represented by a product $\alpha \times \beta$, then the above computation would prove the formula in general. Unfortunately, this is not generally true, so we must resort to another argument.

Recall for a moment how the cap product is computed on the chain level. Given $f \in C^*(X) = \mathrm{Hom}(C_*(X), \mathbb{Z})$, the chain map $C_*(X) \ni a \longmapsto a \frown f \in C_*(X)$ is determined by the composition

$$C_*(X) \xrightarrow{\Delta} C_*(X) \otimes C_*(X) \xrightarrow{f \otimes 1} C_*(X) \tag{2}$$

where Δ is any diagonal approximation (for instance, the Alexander-Whitney diagonal). Recall also the cross product for cochains; given $f \in C^*(X)$ and $g \in C^*(Y)$, the cross product $f \times g \in C^*(X \times Y)$ is the map

$$C_*(X \times Y) \xrightarrow{\theta} C_*(X) \otimes C_*(Y) \xrightarrow{f \otimes g} \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$$

To prove the formula then, consider the diagram

$$\begin{tikzcd}
C_*(X)\otimes C_*(Y) \arrow[rr, "\times"] \arrow[d, "\Delta\otimes\Delta"] & & C_*(X\times Y) \arrow[rr, "\Delta"] & & C_*(X\times Y)\otimes C_*(X\times Y) \arrow[d, "\theta\otimes 1"] \\
C_*(X)\otimes C_*(X)\otimes C_*(Y)\otimes C_*(Y) \arrow[r, leftrightarrow, "\tau"] \arrow[d, "\phi\otimes 1\otimes\psi\otimes 1"] & C_*(X)\otimes C_*(Y)\otimes C_*(X)\otimes C_*(Y) \arrow[rrr, "1\otimes\times"] \arrow[dl, "\phi\otimes\psi\otimes 1\otimes 1"] \arrow[drrr, "\phi\otimes\psi\otimes\times"] & & & C_*(X)\otimes C_*(Y)\otimes C_*(X\times Y) \arrow[d, "\phi\otimes\psi\otimes 1"] \\
C_*(X)\otimes C_*(Y) \arrow[rrrr, "\times"] & & & & C_*(X\times Y)
\end{tikzcd}$$

The map $a\otimes b\longmapsto(a\times b)\frown(\phi\times\psi)$ is the one obtained by starting at the upper left corner and proceeding right two steps and down two steps. The map $a\otimes b\longmapsto(a\frown\phi)\times(b\frown\psi)$ is obtained by starting at the upper left and going down two steps and then right.

The claim is that this diagram commutes up to sign and chain homotopy. (The correct signs could in principle be determined from the diagram, but this is beyond my ability to do without error, and besides we have already worked out the sign above.)

Indeed, the top portion of the diagram consists of chain maps which are natural in X and Y and which have the "obvious" behavior 0 chains, so this top square commutes up to chain homotopy by an acyclic models argument. The bottom portion of the diagram is easily seen to commute up to sign. This completes the proof of the formula.

Before computing the cap products on $S^m\times S^n$, note that the cap product makes sense as a map

$$\frown\colon H_n(X)\otimes H^k(X)\longrightarrow H_{n-k}(X)$$

for all n and k, even if $k>n$. Indeed in the latter case, $n-k<0$ so $H_{n-k}(X)=0$ and the cap product is just the zero map. To justify this claim on the chain level, simply note that if $a\in C_n(X)$ then the image of Δa in $C_*(X)\otimes C_*(X)$ lies in the groups $C_p(X)\otimes C_q(X)$ such that $p+q=n$. We interpret $f\in C^k(X)=\mathrm{Hom}(C_*(X),\mathbb{Z})$ as a chain map which is nonzero only on degree k, so to compute $a\frown f$ as in (2), the only nontrivial mapping occurs on $p=k$, and this is zero if $k>n$ since then we must have $q=n-p=n-k<0$ and $C_p(X)\otimes C_q(X)=C_p(X)\otimes 0=0$.

Denote the generators of $H_0(S^m)$ and $H_0(S^n)$ by 1, and the generators of $H_m(S^m)$ and $H_n(S^n)$ by α and β, respectively. Then denote the generators of $H^0(S^m)$ and $H^0(S^n)$ also by 1 (hopefully no confusion should arise) and generators of $H^m(S^m)$ and $H^n(S^n)$ respectively by ϕ and ψ. Cap products between the elements for S^m satisfy

$$\begin{aligned} \alpha\frown\phi&=1 \qquad & \alpha\frown 1&=\alpha\\ 1\frown\phi&=0 \qquad & 1\frown 1&=1, \end{aligned}$$

and similarly for S^n.

The homology and cohomology groups of $S^m \times S^n$, $m \neq n$, are given by

$$H_k(S^m \times S^n) = \begin{cases} \mathbb{Z}\langle 1 \times 1 \rangle & k = 0 \\ \mathbb{Z}\langle \alpha \times 1 \rangle & k = m \\ \mathbb{Z}\langle 1 \times \beta \rangle & k = n \\ \mathbb{Z}\langle \alpha \times \beta \rangle & k = m + n \end{cases} \qquad H^k(S^m \times S^n) = \begin{cases} \mathbb{Z}\langle 1 \times 1 \rangle & k = 0 \\ \mathbb{Z}\langle \phi \times 1 \rangle & k = m \\ \mathbb{Z}\langle 1 \times \psi \rangle & k = n \\ \mathbb{Z}\langle \phi \times \psi \rangle & k = m + n \end{cases}$$

and zero otherwise. When $n = m$ the groups are given by

$$H_k(S^m \times S^m) = \begin{cases} \mathbb{Z}\langle 1 \times 1 \rangle & k = 0 \\ \mathbb{Z}\langle \alpha \times 1, 1 \times \beta \rangle & k = m \\ \mathbb{Z}\langle \alpha \times \beta \rangle & k = 2m \end{cases} \quad H^k(S^m \times S^m) = \begin{cases} \mathbb{Z}\langle 1 \times 1 \rangle & k = 0 \\ \mathbb{Z}\langle \phi \times 1, 1 \times \psi \rangle & k = m \\ \mathbb{Z}\langle \phi \times \psi \rangle & k = 2m \end{cases}$$

and zero otherwise. Using the formula, the nontrivial cap products on $S^m \times S^n$ (in either case $m \neq n$ or $m = n$) are therefore determined by the following multiplication table for generators:

$\frown$	$\phi \times \psi$	$\phi \times 1$	$1 \times \psi$	1×1
$\alpha \times \beta$	1×1	$1 \times \beta$	$(-1)^{mn}\alpha \times 1$	1×1
$\alpha \times 1$	0	1×1	0	$\alpha \times 1$
$1 \times \beta$	0	0	1×1	$1 \times \beta$
1×1	0	0	0	1×1

Note: don't be confused by the tempting but false(!) formula $\alpha \times 1 \stackrel{?}{=} 1 \times \beta$ when $m = n$. The two elements α and β are distinct and not comparable. The general graded commutativity result for the cross product states that $\alpha \times \beta = (-1)^{|\alpha||\beta|} T^*(\beta \times \alpha)$ where $T : X \times Y \longrightarrow Y \times X$ is the transposition. So the correct (though not actually useful) statement in this case is

$$H_m(S^m_a \times S^m_b) \ni \alpha \times 1 \cong 1 \times \alpha \in H_m(S^m_b \times S^m_a)$$

where we have been careful to label the factors of S^m to keep them distinct.

$\square$