Math 2420 – Problem Set 3, due Monday 3/12.

Update 3/12: There is some sign involved in the cap/cross product formula in problem 3. Thanks to Yilong for pointing this out.

Problem 1. Show that manifolds M and N are orientable if and only if $M \times N$ is orientable. Show also that if $\alpha \in H_m(M; R)$ and $\beta \in H_n(N; R)$ are fundamental classes, then $\alpha \times \beta \in H_{m+n}(M \times N; R)$ is a fundamental class.

Problem 2. For a (locally compact Hausdorff) space X let X^+ denote the one-point compactification. If the added point $\infty \in X^+$ has a neighborhood which is a cone with ∞ as a cone point (a neighborhood deformation retract) show that the evident map

$$H_c^n(X; G) \longrightarrow H^n(X^+, \infty; G)$$

is an isomorphism for all n.

Problem 3. Prove that, for $\phi \in H^*(X)$, $\psi \in H^*(Y)$, $a \in H_*(X)$ and $b \in H_*(Y)$, the cap and cross products are related by

$$(a \times b) \frown (\phi \times \psi) = (-1)^*(a \frown \phi) \times (b \frown \psi),$$

with a sign that you should determine. Use this to compute all cap products in homology and cohomology of $S^m \times S^n$, where m and n may be equal.