Problem 1. (a) For $H \leq G$ and a fixed $g \in G$ show that gHg^{-1} is a subgroup of G with the same order as H.

(b) Conclude that if H is the unique subgroup of order n then H is normal.

Solution. (a) That gHg^{-1} is a subgroup can be verified by the subgroup criterion. Indeed, suppose gh_1g^{-1} and gh_2g^{-1} are two elements in gHg^{-1}. Then

$$gh_1g^{-1}\left(gh_2g^{-1}\right)^{-1} = gh_1g^{-1}gh_2^{-1}g^{-1} = gh_1h_2^{-1}g^{-1} \in gHg^{-1}$$

since $h_1h_2^{-1} \in H$.

To see that H and gHg^{-1} have the same order, observe that

$$f : H \longrightarrow gHg^{-1}, \qquad f(h) = ghg^{-1}$$

is a bijection. Indeed, it is manifestly surjective, and if $gh_1g^{-1} = gh_2g^{-1}$ then cancellation implies $h_1 = h_2$. (In fact f is a homomorphism, but this is not needed here.)

(b) Assume H is the unique subgroup with order n. Then for any $g \in G$, gHg^{-1} is a subgroup of order n by part (a), hence it must be H:

$$gHg^{-1} = H, \qquad \text{for all } g \in G,$$

but this is precisely the statement that H is normal.

□

Problem 2. Suppose H and K are subgroups of G, and that the greatest common divisor $(|H|, |K|) = 1$. Then $H \cap K = \{1\}$.

Solution. Since intersections of subgroups are subgroups, it follows that $H \cap K \leq H$ and $H \cap K \leq K$. Then by Lagrange's Theorem, $|H \cap K|$ must divide $|H|$ and $|K|$. Since $(|H|, |K|) = 1$ it follows that $|H \cap K| = 1$ and therefore that $H \cap K = \{1\}$. □

Problem 3. If $H \leq K \leq G$, then $|G : K|\ |K : H| = |G : H|$.

Solution. By definition $|G : H|$ is the number of elements in the set $\{gH : g \in G\}$ of cosets of H in G. Recall that G is partitioned into these cosets, which are the equivalence classes with respect to the relation $g_1 \sim_H g_2 \iff g_1g_2^{-1} \in H$.

The idea is now to *partition the set of cosets* $\{gH : g \in G\}$ with respect to the equivalence relation

$$g_1H \sim g_2H \iff g_1g_2^{-1} \in K.$$

We denote the equivalence class of gH with respect to this equivalence relation by $[gH]$, so

$$[gH] = \{g'H : g'H \sim gH\}.$$

There are $|G : K|$ distinct equivalence classes, since each may be labeled by an equivalence class of $g \in G$ with respect to the equivalence relation $g_1 \sim_K g_2 \iff g_1g_2^{-1} \in K$, which is none other than the set of cosets $\{gK : g \in G\}$.

Each equivalence class has the same size, since if $[g_1H]$ and $[g_2H]$ are two equivalence classes, the map

$$[g_1H] \ni gH \longmapsto g'H \in [g_2H], \qquad g' = gg_1^{-1}g_2$$

is a bijection, with inverse

$$g'H \longmapsto gH, \qquad g = g'g_2^{-1}g_1.$$

Note that this works since $g'g_2^{-1} = gg_1^{-1}g_2g_2^{-1} = gg_1^{-1}$ so $g'g_2^{-1} \in K \iff gg_1^{-1} \in K$.

The size of any equivalence class is therefore equal to the size of the particular equivalence class

$$[kH] = \{gH : g \in K\} = \text{cosets of } H \text{ in } K$$

which is $|[kH]| = |K : H|$. We therefore conclude that

$$|G : H| = |G : K|\ |K : H|.$$

□

Problem 4. Let $|G| < \infty$, $H \leq G$, $N \trianglelefteq G$, and suppose that $(|H|, |G : N|) = 1$. Then $H \leq N$.

Proof. The idea is to show that because $|H|$ and $|G : N|$ are coprime, H must to to 1 in the quotient $\pi : G \longrightarrow G/N$ (this is equivalent to $H \leq N$).

So consider the image $\pi(H)$. We certainly have $\pi(H) \leq G/N$ since the image of a subgroup under a homomorphism is a subgroup, thus $|\pi(H)|$ divides $|G/N| = |G : N|$ by Lagrange's Theorem.

On the other hand, $|\pi(H)|$ must divide $|H|$ since $\pi(H) \cong H/\ker(\pi : H \to G/N)$, so that $|H| = |\pi(H)|\,|\ker(\pi : H \to G/N)|$ (in fact the kernel here is $H \cap N \trianglelefteq H$, but that is not strictly needed).

Since $(|H|, |G : N|) = 1$ it now follows that $(|\pi(H)|, |G : N|) = 1$, so

$$|\pi(H)| = 1 \implies \pi(H) = \{1\} \implies H \leq N.$$

□

Problem 5 (The 4th isomorphism Theorem)**.** Let $N \trianglelefteq G$. Then subgroups of $\overline{G} = G/N$ are in bijection with subgroups of G which contain N via

$$\overline{A} \leq G/N \iff \overline{A} = \pi(A), \quad N \leq A \leq G,$$

where $\pi : G \longrightarrow G/N$ denotes the canonical projection. Furthermore, if $N \leq A$ and $N \leq B$, then

(1) $A \leq B$ if and only if $\overline{A} \leq \overline{B}$.
(2) If $A \leq B$ then $|B : A| = \left|\overline{B} : \overline{A}\right|$.
(3) $\overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B}\rangle$.
(4) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
(5) $A \trianglelefteq G$ if and only if $\overline{A} \trianglelefteq \overline{G}$.

Proof. If $N \leq A \leq G$, then $\pi(A) = A/N$ is a subgroup of $\overline{G}$. In the other direction, if $\overline{A} \leq \overline{G}$, then $A = \pi^{-1}(\overline{A})$ is a subgroup of G (it is a general fact that the inverse image of a subgroup with respect to a homomorphism is a subgroup), and $N \leq A$ since $N = \pi^{-1}(1)$ and $1 \in \overline{A}$. Furthermore, the maps sending A to $\overline{A} = \pi(A)$ and $\overline{A}$ to $\pi^{-1}(A)$ are clearly inverses, so these subgroups are in bijection.

To prove property (1), suppose that $A \leq B$. Then since every a in A is also in B it follows that

$$A/N = \{aN : a \in A\} \leq B/N = \{bN : b \in B\}.$$

Conversely, suppose that $\overline{A} \leq \overline{B}$. Then every $aN \in \overline{A}$ is also in $\overline{B}$, which means that $aN = bN$ for some $b \in B$, and it follows that $a = bn$ for some $n \in N$. But $bn \in B$ since $N \leq B$, so $a \in B$. Since $a \in A$ was arbitrary, we conclude $A \leq B$.

For (2), we construct a bijection between the cosets $\{bA : b \in B\}$ of A in B and the cosets $\left\{bN\overline{A} : bN \in \overline{B}\right\}$ of $\overline{A}$ in $\overline{B}$, via

$$bA \longmapsto bN\overline{A}. \tag{1}$$

To see that this is well-defined, suppose that $b_1A = b_2A$. Then $b_1 = b_2a$ for some $a \in A$ and it follows that

$$b_1N\overline{A} = b_2aN\overline{A} = b_2Na\overline{A} = b_2N\overline{A}.$$

In the second equality we used the fact that N is normal, so $aNa^{-1} = N \iff aN = Na$, and the third equality follows from the fact that, for every $a'N \in \overline{A}$, $a(a'N) = aa'N \in \overline{A}$ and vice versa. The map (1) is clearly surjective. To see that it is injective, suppose that $b_1N\overline{A} = b_2N\overline{A}$. Then

$$b_1N = b_2NaN, \quad \text{for some } aN \in \overline{A},$$

and by normality it follows that

$$b_1N = b_2aNN = b_2aN,$$

so $b_1 = b_2an$ for some $n \in N$. Since $N \leq A$, this means $b_1 = b_2a'$ where $a' = an \in A$ so that $b_1A = b_2A$.

For (3), recall that $D = \langle A, B\rangle$ is the smallest subgroup containing A and B. From (1) it follows that $\overline{D} = D/N$ contains both $\overline{A}$ and $\overline{B}$. If $\overline{D}'$ were a smaller subgroup containing $\overline{A}$ and $\overline{B}$, it would follow, again from (1), that $\pi^{-1}(\overline{D}')$ was a strictly smaller subgroup containing A and B, which is a contradiction. We conclude that

$$\overline{D} = \overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B}\rangle.$$

Part (4) is equivalent to the statement that $\pi^{-1}(\overline{A} \cap \overline{B}) = A \cap B$. To see one direction, note that, for every $g \in A \cap B$, $\pi(g) \in \overline{A} \cap \overline{B}$, so $A \cap B \subseteq \pi^{-1}(\overline{A} \cap \overline{B})$. For the other direction, observe that $\overline{A} \cap \overline{B} \leq \overline{A}$ and $\overline{A} \cap \overline{B} \leq \overline{B}$, so by part (1) it follows that $\pi^{-1}(\overline{A} \cap \overline{B}) \leq A \cap B$.

Finally, for part (5), first assume that $A \trianglelefteq G$, and consider an arbitrary $aN \in \overline{A}$ and $gN \in \overline{G}$. Then

$$(gN)(aN)(gN)^{-1} = gN\,aN\,g^{-1}N = gag^{-1}N = a'N \in \overline{A} \tag{2}$$

for some $a' \in A$, and it follows that $\overline{A} \trianglelefteq \overline{G}$. Conversely, suppose that $\overline{A} \trianglelefteq \overline{G}$. For arbitrary $a \in A$ and $g \in G$, the assumption means that (2) holds, so in particular $gag^{-1} = a'n \in A$ for some $a' \in A$, $n \in N$. Since $N \leq A$, $a'n \in A$, so that $A \trianglelefteq G$. □

Problem 6. Let A and B be two groups, with normal subgroups $C \trianglelefteq A$ and $D \trianglelefteq B$, respectively. Then $(C \times D) \trianglelefteq (A \times B)$ and

$$(A \times B)/(C \times D) \cong (A/C) \times (B/D).$$

Proof. Recall that $A \times B$ is a group with respect to componentwise multiplication: $(a, b) \cdot (a', b') = (aa', bb')$ and inverses: $(a, b)^{-1} = (a^{-1}, b^{-1})$.. Let $(c, d) \in C \times D$ and $(a, b) \in A \times B$. Then

$$(a, b)\,(c, d)\,(a, b)^{-1} = (aca^{-1}, bdb^{-1}) \in C \times D$$

by normality of C in A and D in B. Thus $C \times D \trianglelefteq A \times B$.

Note that cosets of $C \times D$ in $A \times B$ have the form $(a, b)(C \times D)$ for various $(a, b) \in A \times B$. Define a surjective map $\phi : (A \times B)/(C \times D) \longrightarrow (A/C) \times (B/D)$ by

$$\phi\big((a, b)(C \times D)\big) = (aC, bD) \in (A/C) \times (B/D).$$

ϕ is a homomorphism since

$$\begin{aligned}\phi\big((a, b)(C \times D)\,(a', b')(C \times D)\big) &= \phi\big((aa', bb')(C \times D)\big)\\ &= (aa'C, bb'D)\\ &= (aC, bD)\,(a'C, b'D)\\ &= \phi\big((a, b)(C \times D)\big)\phi\big((a', b')(C \times D)\big).\end{aligned}$$

Finally, ϕ is injective since

$$(aC, bD) = (a'C, b'D) \iff a = a'c, \quad b = b'd$$

for some $c \in C$ and $d \in D$, and it follows that

$$(a, b)(C \times D) = (a'c, b'd)(C \times D) = (a', b')\,(c, d)(C \times D) = (a', b')(C \times D).$$ □